Fixed Point Theory, 15(2014), No. 1, 33-42
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A TOPOLOGICAL PROPERTY OF SOLUTION SETS OF SEMILINEAR DIFFERENTIAL INCLUSIONS

AURELIAN CERNEA

Faculty of Mathematics and Informatics, University of Bucharest
Academiei 14, 010014 Bucharest, Romania
E-mail: acernea@fmi.unibuc.ro

Abstract. We consider a Cauchy problem for a semilinear differential inclusion involving a non-convex set-valued map and we prove that the set of selections corresponding to the solutions of the problem considered is a retract of the space of integrable functions on unbounded interval. A similar result is provided for a class of second-order differential inclusions.
Key Words and Phrases: differential inclusion, decomposable set, retract.
2010 Mathematics Subject Classification: 34A60, 47H04.

1. Introduction

This paper is concerned with the following semilinear differential inclusion

$$x' \in Ax + F(t,x), \quad x(0) = x_0, \tag{1.1}$$

where X is a real separable Banach space, $\mathcal{P}(X)$ is the family of all subsets of X, $F(.,.) : [0,\infty) \times X \to \mathcal{P}(X)$ and A is the infinitesimal generator of a strongly continuous semigroup $\{G(t);\ t \geq 0\}$ on X.

Existence results and qualitative properties of the mild solutions of problem (1.1) may be found in [5,6,9,11,13,15] etc.. In [8] we proved that the solution set of problem (1.1) is arcwise connected when the set-valued map is Lipschitz in the second variable and the problem is defined on a bounded interval. The aim of this paper is to establish a more general topological property of the solution set of problem (1.1). Namely, we prove that the set of selections of the set-valued map F that correspond to the solutions of problem (1.1) is a retract of $L^1_{loc}([0,\infty),X)$. The result is essentially based on Bressan and Colombo results ([3, 14]) concerning the existence of continuous selections of lower semicontinuous set-valued maps with decomposable values.

A similar result is valid for second-order differential inclusions of the form

$$x'' \in Ax + F(t,x), \quad x(0) = x_0, \quad x'(0) = y_0, \tag{1.2}$$

where F is as above and A is the infinitesimal generator of a strongly continuous cosine family of operators $\{C(t);\ t \geq 0\}$ on X. Several qualitative properties and existence results concerning mild solutions for the Cauchy problem (1.2) can be found in [1, 2, 7, 8, 9] etc..

Work supported by the CNCS grant PN-II-ID-PCE-2011-3-0198.

We note that in the classical case of differential inclusions topological properties of solution set are obtained using various methods and tools ([4, 10, 16-18] etc.). The results in the present paper extends to semilinear differential inclusions of the form (1.1) and (1.2) the main result in [16] obtained in the case of classical differential inclusions.

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove our main result.

2. Preliminaries

Let $T > 0$, $I := [0,T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with the norm $|.|$. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $cl(A)$ the closure of A.

The distance between a point $x \in X$ and a subset $A \subset X$ is defined as usual by $d(x,A) = \inf\{|x-a|; a \in A\}$. We recall that Pompeiu-Hausdorff distance between the closed subsets $A, B \subset X$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, $d^*(A,B) = \sup\{d(a,B);\, a \in A\}$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x : I \to X$ endowed with the norm $|x|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x : I \to X$ endowed with the norm $|x|_1 = \int_0^T |x(t)|dt$.

We recall first several preliminary results we shall use in the sequel.

A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u, v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$.

We denote by $\mathcal{D}(I,X)$ the family of all decomposable closed subsets of $L^1(I,X)$.

Next (S,d) is a separable metric space; we recall that a set-valued map $G : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S;\ G(s) \subset C\}$ is closed.

Lemma 2.1. ([3]) *Let $F^* : I \times S \to \mathcal{P}(X)$ be a closed-valued $\mathcal{L}(I) \otimes \mathcal{B}(S)$-measurable set-valued map such that $F^*(t,.)$ is l.s.c. for any $t \in I$.*

Then the set-valued map $G : S \to \mathcal{D}(I,X)$ defined by

$$G(s) = \{v \in L^1(I,X); \quad v(t) \in F^*(t,s) \quad a.e.\ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p : S \to L^1(I,X)$ such that

$$d(0, F^*(t,s)) \le p(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

Lemma 2.2. ([3]) *Let $G : S \to \mathcal{D}(I,X)$ be a l.s.c. set-valued map with closed decomposable values and let $\phi : S \to L^1(I,X)$, $\psi : S \to L^1(I,\mathbf{R})$ be continuous such that the set-valued map $H : S \to \mathcal{D}(I,X)$ defined by*

$$H(s) = cl\{v(.) \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad a.e.\ (I)\}$$

has nonempty values.

Then H has a continuous selection, i.e. there exists a continuous mapping $h: S \to L^1(I, X)$ such that $h(s) \in H(s) \quad \forall s \in S$.

In what follows X is a real separable Banach space with norm $|.|$, and with the corresponding metric $d(.,.)$. We consider $\{G(t)\}_{t\geq 0} \subset L(X,X)$ a strongly continuous semigroup of bounded linear operators from X to X having the infinitesimal generator A and a set valued map $F(.,.)$ defined on $[0,\infty) \times X$ with nonempty closed subsets of X, which define the following differential inclusion

$$x' \in Ax + F(t,x) \quad x(0) = x_0. \tag{2.1}$$

It is well known that, in general, the Cauchy problem

$$x' = Ax + f(t,x), \quad x(0) = x_0 \tag{2.2}$$

may not have a classical solution and that a way to overcome this difficulty is to look for continuous solutions of the integral equation

$$x(t) = G(t)x_0 + \int_0^t G(t-u)f(u, x(u))du$$

This is why the concept of the mild solution is convenient for solving (2.1).

A continuous mapping $x(.) \in C([0,\infty), X)$ is called a *mild solution* of (2.1) if there exists a (Bochner) integrable function $f(.) \in L^1_{loc}([0,\infty), X)$ such that

$$f(t) \in F(t, x(t)) \quad a.e.\ [0,\infty), \tag{2.3}$$

$$x(t) = G(t)x_0 + \int_0^t G(t-u)f(u)du \quad \forall t \in [0,\infty), \tag{2.4}$$

i.e., $f(.)$ is a (Bochner) integrable selection of the set-valued map $F(., x(.))$ and $x(.)$ is the mild solution of the initial value problem

$$x' = Ax + f(t) \quad x(0) = x_0. \tag{2.5}$$

We shall use the following notations

$$\mathcal{S}^1(x_0) = \{x(.); \quad x(.) \text{ is a mild solution of (1.1)}\}, \tag{2.6}$$

$$\begin{aligned} \mathcal{T}^1(x_0) = \{ & f \in L^1_{loc}([0,\infty), X);\ f(t) \in F(t, G(t)x_0 + \\ & + \textstyle\int_0^t G(t-u)f(u)du) \quad a.e.\ [0,\infty)\}. \end{aligned} \tag{2.7}$$

Denote by $B(X)$ the Banach space of bounded linear operators from X into X. We recall that a family $\{C(t);\ t \in \mathbf{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following conditions are satisfied

(i) $C(0) = I$, where I is the identity operator in X,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)\ \forall t, s \in \mathbf{R}$,

(iii) the map $t \to C(t)x$ is strongly continuous $\forall x \in X$.

The strongly continuous sine family $\{S(t);\ t \in \mathbf{R}\}$ associated to a strongly continuous cosine family $\{C(t);\ t \in \mathbf{R}\}$ is defined by

$$S(t)x := \int_0^t C(s)x ds, \quad x \in X, t \in \mathbf{R}.$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t);\ t \in \mathbf{R}\}$ is defined by

$$Ax = (\frac{d^2}{dt^2})C(t)x|_{t=0}.$$

Fore more details on strongly continuous cosine and sine family of operators we refer to [12, 19].

In what follows A is infinitesimal generator of a cosine family $\{C(t);\ t \in \mathbf{R}\}$ and $F(.,.) : I \times X \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values, which define the following Cauchy problem associated to a second-order differential inclusion

$$x'' \in Ax + F(t,x), \quad x(0) = x_0, \quad x'(0) = x_1. \tag{2.8}$$

A continuous mapping $x(.) \in C([0,\infty), X)$ is called a *mild solution* of problem (2.8) if there exists a (Bochner) integrable function $f(.) \in L^1_{loc}([0,\infty), X)$ such that

$$f(t) \in F(t, x(t)) \quad a.e.\ [0,\infty) \tag{2.9}$$

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-u)f(u)du \quad \forall t \in [0,\infty), \tag{2.10}$$

i.e., $f(.)$ is a (Bochner) integrable selection of the set-valued map $F(., x(.))$ and $x(.)$ is the mild solution of the Cauchy problem

$$x'' = Ax + f(t) \quad x(0) = x_0, \quad x'(0) = x_1. \tag{2.11}$$

We make the following notations

$$\mathcal{S}^2(x_0, x_1) = \{x(.); \quad x(.) \text{ is a mild solution of (1.2)}\}, \tag{2.12}$$

$$\begin{aligned} \mathcal{T}^2(x_0, x_1) = \{f \in L^1_{loc}([0,\infty), X); \quad f(t) \in F(t, C(t)x_0 + S(t)x_1 + \\ + \textstyle\int_0^t S(t-u)f(u)du) \quad a.e.\ [0,\infty)\}. \end{aligned} \tag{2.13}$$

3. The main results

In order to prove our topological properties of the solution set of problems (1.1) and (1.2) we need the following hypotheses.

Hypothesis 3.1. i) $F(.,.) : [0,\infty) \times X \to \mathcal{P}(X)$ has nonempty compact values and is $\mathcal{L}([0,\infty)) \otimes \mathcal{B}(X)$ measurable.

ii) There exists $L \in L^1_{loc}([0,\infty), \mathbf{R})$ such that, for almost all $t \in [0,\infty)$, $F(t,.)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x), F(t,y)) \le L(t)|x-y| \quad \forall x, y \in X.$$

iii) There exists $p \in L^1_{loc}([0,\infty), \mathbf{R})$ such that

$$d_H(\{0\}, F(t,0)) \le p(t) \quad a.e.\ [0,\infty).$$

We consider first the semilinear differential inclusion (1.1). Let $M \ge 1$ be such that $|G(t)| \le M \quad \forall t \in [0,\infty)$.

Take $I = [0,T]$ and we make the notations

$$\tilde{u}(t) = G(t)x_0 + \int_0^t G(t-s)u(s)ds, \quad u \in L^1(I,X) \tag{3.1}$$

and

$$p_0(u)(t) = |u(t)| + p(t) + L(t)|\tilde{u}(t)|, \quad t \in I \tag{3.2}$$

Let us note that

$$d(u(t), F(t, \tilde{u}(t)) \leq p_0(u)(t) \quad a.e.\ (I) \tag{3.3}$$

and, since for any $u_1, u_2 \in L^1(I, X)$

$$|p_0(u_1) - p_0(u_2)|_1 \leq (1 + M \int_0^T L(s)ds|)|u_1 - u_2|_1$$

the mapping $p_0 : L^1(I, X) \to L^1(I, X)$ is continuous.

Also define

$$\mathcal{T}_I(x_0) = \{f \in L^1(I, X); \quad f(t) \in F(t, G(t)x_0 + \int_0^t G(t-s)f(s)ds) \quad a.e.\ (I)\}.$$

Proposition 3.2. *Assume that Hypothesis 3.1 is satisfied and let $\phi : L^1(I, X) \to L^1(I, X)$ be a continuous map such that $\phi(u) = u$ for all $u \in \mathcal{T}_I(x_0)$. For $u \in L^1(I, X)$, we define*

$$\Psi(u) = \{u \in L^1(I, X); \quad u(t) \in F(t, \widetilde{\phi(u)}(t)) \quad a.e.\ (I)\},$$

$$\Phi(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_I(x_0), \\ \Psi(u) & \text{otherwise.} \end{cases}$$

Then the set-valued map $\Phi : L^1(I, X) \to \mathcal{P}(L^1(I, X))$ is lower semicontinuous with closed decomposable and nonempty values.

Proof. According to (3.3), Lemma 2.1 and the continuity of p_0 we obtain that Ψ has closed decomposable and nonempty values and the same holds for the set-valued map Φ.

Let $C \subset L^1(I, X)$ be a closed subset, let $\{u_m\}_{m\in\mathbf{N}}$ converges to some $u_0 \in L^1(I, X)$ and $\Phi(u_m) \subset C$, for any $m \in \mathbf{N}$. Let $v_0 \in \Phi(u_0)$ and for every $m \in \mathbf{N}$ consider a measurable selection v_m from the set-valued map $t \to F(t, \widetilde{\phi(u_m)}(t))$ such that $v_m = u_m$ if $u_m \in \mathcal{T}_I(x_0)$ and

$$|v_m(t) - v_0(t)| = d(v_0(t), F(t, \widetilde{\phi(u_m)}(t)) \quad a.e.\ (I)$$

otherwise. One has

$$|v_m(t) - v_0(t)| \leq$$

$$\leq d_H(F(t, \widetilde{\phi(u_m)}(t)), F(t, \widetilde{\phi(u_0)}(t))) \leq L(t)|\widetilde{\phi(u_m)}(t) - \widetilde{\phi(u_0)}(t)|$$

hence

$$|v_m - v_0|_1 \leq M \int_0^T L(s)ds.|\widetilde{\phi(u_m)} - \widetilde{\phi(u_0)}|_1.$$

Since $\phi : L^1(I, X) \to L^1(I, X)$ is continuous, it follows that v_m converges to v_0 in $L^1(I, X)$. On the other hand, $v_m \in \Phi(u_m) \subset C\ \forall m \in \mathbf{N}$ and since C is closed we infer that $v_0 \in C$. Hence $\Phi(u_0) \subset C$ and Φ is lower semicontinuous.

In what follows we shall use the following notations

$$I_k=[0,k],\quad k\geq 1,\quad |u|_{1,k}=\int_0^k |u(t)|dt,\quad u\in L^1(I_k,X).$$

We are able now to prove the main result of this paper.

Theorem 3.3. *Consider A the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{G(t)\}_{t\geq 0}$ on the real separable Banach space X, assume that Hypothesis 3.1 is satisfied, let $x_0\in X$ and let $\mathcal{T}^1(x_0)$ be the selection set defined in (2.7).*

Then there exists a continuous mapping $G:L^1_{loc}([0,\infty),X)\to L^1_{loc}([0,\infty),X)$ such that

(i) $G(u)\in \mathcal{T}^1(x_0),\quad \forall u\in L^1_{loc}([0,\infty),X)$,

(ii) $G(u)=u,\quad \forall u\in \mathcal{T}^1(x_0)$.

Proof. We shall prove that for every $k\geq 1$ there exists a continuous mapping $g^k: L^1(I_k,X)\to L^1(I_k,X)$ with the following properties

(I) $g^k(u)=u,\quad \forall u\in \mathcal{T}_{I_k}(x_0)$

(II) $g^k(u)\in \mathcal{T}_{I_k}(x_0),\quad \forall u\in L^1(I_k,X)$

(III) $g^k(u)(t)=g^{k-1}(u|_{I_{k-1}})(t),\quad \forall t\in I_{k-1}$

If the sequence $\{g^k\}_{k\geq 1}$ is constructed, we define $G:L^1_{loc}([0,\infty),X)\to L^1_{loc}([0,\infty),X)$ by

$$G(u)(t)=g^k(u|_{I_k})(t),\quad \forall k\geq 1$$

From (III) and the continuity of each $g^k(.)$ it follows that $G(.)$ is well defined and continuous. Moreover, for each $u\in L^1_{loc}([0,\infty),X)$, according to (II) we have

$$G(u)|_{I_k}(t)=g^k(u|_{I_k})(t),\quad g^k(u|_{I_k})\in \mathcal{T}_{I_k}(x_0),\quad \forall k\geq 1$$

and thus $G(u)\in \mathcal{T}(x_0)$.

Fix $\varepsilon>0$ and for $m\geq 0$ set $\varepsilon_m=\frac{m+1}{m+2}\varepsilon$. For $u\in L^1(I_1,X)$ and $m\geq 0$ define $m(t)=\int_0^t L(s)ds$,

$$p_0^1(u)(t)=|u(t)|+p(t)+L(t)|\tilde{u}(t)|,\ \ t\in I_1$$

and

$$p^1_{m+1}(u)(t)=M^{m+1}\int_0^t p_0^1(u)(s)\frac{(m(t)-m(s))^m}{m!}ds+M^m\frac{(m(t))^m}{m!}\varepsilon_{m+1}.$$

By the continuity of the map $p_0^1(.)=p_0(.)$, already proved, we obtain that $p^1_m: L^1(I_1,X)\to L^1(I_1,X)$ is continuous.

We define $g^1_0(u)=u$ and we shall prove that for any $m\geq 1$ there exists a continuous map $g^1_m:L^1(I_1,X)\to L^1(I_1,X)$ that satisfies

$$(a_1)\qquad g^1_m(u)=u,\quad \forall u\in \mathcal{T}_{I_1}(x_0),$$

$$(b_1)\qquad g^1_m(u)(t)\in F(t,\widetilde{g^1_{m-1}(u)}(t))\quad a.e.\ (I_1),$$

$$(c_1)\qquad |g^1_1(u)(t)-g^1_0(u)(t)|\leq p^1_0(u)(t)+\varepsilon_0\quad a.e.\ (I_1),$$

(d_1) $\qquad |g_m^1(u)(t) - g_{m-1}^1(t)| \leq L(t)p_{m-1}^1(u)(t) \quad a.e.\ (I_1), \quad m \geq 2.$

For $u \in L^1(I_1, X)$, we define

$$\Psi_1^1(u) = \{v \in L^1(I_1, X);\ \ v(t) \in F(t, \widetilde{u}(t))\ \ a.e.(I_1)\},$$

$$\Phi_1^1(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_1}(x_0), \\ \Psi_1^1(u) & \text{otherwise.} \end{cases}$$

and by Proposition 3.2 (with $\phi(u) = u$) we obtain that $\Phi_1^1 : L^1(I_1, X) \to \mathcal{D}(I_1, X)$ is lower semicontinuous. Moreover, due to (3.3) the set

$$H_1^1(u) = cl\{v \in \Phi_1^1(u);\quad |v(t) - u(t)| < p_0^1(u)(t) + \varepsilon_0 \quad a.e.\ (I_1)\}$$

is not empty for any $u \in L^1(I_1, X)$. So applying Lemma 2.2, we find a continuous selection g_1^1 of H_1^1 that satisfies (a_1)-(c_1).

Suppose we have already constructed $g_i^1(.)$, $i = 1, \dots m$ satisfying (a_1)-(d_1). Then from (b_1), (d_1) and Hypothesis 3.1 we get

$$\begin{array}{l} d(g_m^1(u)(t), F(t, \widetilde{g_m^1(u)}(t)) \leq L(t)(|\widetilde{g_{m-1}^1(u)}(t) - \widetilde{g_m^1(u)}(t)| \leq \\ L(t)\int_0^T ML(s)p_m^1(u)(s)ds = L(t)(p_{m+1}^1(u)(t) - r_m^1(t)) < L(t)p_{m+1}^1(u)(t), \end{array} \tag{3.4}$$

where $r_m^1(t) := M^m \frac{(m(t))^m}{m!}(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $u \in L^1(I_1, X)$, we define

$$\Psi_{m+1}^1(u) = \{v \in L^1(I_1, X);\ \ v(t) \in F(t, \widetilde{g_m^1(u)}(t)) \quad a.e.\ (I_1)\},$$

$$\Phi_{m+1}^1(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_1}(x_0), \\ \Psi_{m+1}^1(u) & \text{otherwise.} \end{cases}$$

We apply Proposition 3.2 (with $\phi(u) = g_m^1(u)$) and obtain that $\Phi_{m+1}^1(.)$ is lower semicontinuous with closed decomposable and nonempty values. Moreover, by (3.4), the set

$$H_{m+1}^1(u) = cl\{v \in \Phi_{m+1}^1(u); |v(t) - g_{m+1}^1(u)(t)| < L(t)p_{m+1}^1(u)(t)\ a.e.\ (I_1)\}$$

is nonempty for any $u \in L^1(I_1, X)$. With Lemma 2.2, we find a continuous selection g_{m+1}^1 of H_{m+1}^1, satisfying (a_1)-(d_1).

Therefore we obtain that

$$|g_{m+1}^1(u) - g_m^1(u)|_{1,1} \leq \frac{(Mm(1))^m}{m!}(M|p_0^1(u)|_{1,1} + \varepsilon)$$

and this implies that the sequence $\{g_m^1(u)\}_{m \in \mathbf{N}}$ is a Cauchy sequence in the Banach space $L^1(I_1, X)$. Let $g^1(u) \in L^1(I_1, X)$ be its limit. The function $s \to |p_0^1(u)|_{1,1}$ is continuous, hence it is locally bounded and the Cauchy condition is satisfied by $\{g_m^1(u)\}_{m \in \mathbf{N}}$ locally uniformly with respect to u. Hence the mapping $g^1(.) : L^1(I_1, X) \to L^1(I_1, X)$ is continuous.

From (a_1) it follows that $g^1(u) = u, \quad \forall u \in \mathcal{T}_{I_1}(x_0)$ and from (b_1) and the fact that F has closed values we obtain that

$$g^1(u)(t) \in F(t, \widetilde{g^1(u)}(t)), \quad a.e.\ (I_1) \quad \forall u \in L^1(I_1, X).$$

In the next step of the proof we suppose that we have already constructed the mappings $g^i(.) : L^1(I_i, X) \to L^1(I_i, X),\ i = 2, ..., k-1$ with the properties (I)-(III) and we shall construct a continuous map $g^k(.) : L^1(I_k, X) \to L^1(I_k, X)$ satisfying (I)-(III).

Let $g_0^k : L^1(I_k, X) \to L^1(I_k, X)$ be defined by

$$g_0^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}} + u(t)\chi_{I_k \setminus I_{k-1}}(t) \tag{3.5}$$

Let us note, first, that $g_0^k(.)$ is continuous. Indeed, if $u_0, u \in L^1(I_k, X)$ one has

$$|g_0^k(u) - g_0^k(u_0)|_{1,k} \leq |g^{k-1}(u|_{I_{k-1}}) - g^{k-1}(u_0|_{I_{k-1}})|_{1,k-1} + \int_{k-1}^{k} |u(t) - u_0(t)|dt$$

So, using the continuity of $g^{k-1}(.)$ we get the continuity of $g_0^k(.)$.

On the other hand, since $g^{k-1}(u) = u$, $\forall u \in \mathcal{T}_{I_{k-1}}(x_0)$ from (3.5) it follows that

$$g_0^k(u) = u, \quad \forall u \in \mathcal{T}_{I_k}(x_0).$$

For $u \in L^1(I_k, X)$, we define

$$\Psi_1^k(u) = \{w \in L^1(I_k, X); \;\; w(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}}(t) + v(t)\chi_{I_k \setminus I_{k-1}}(t),$$
$$v(t) \in F(t, \widetilde{g_0^k(u)(t)}) \quad a.e. \; ([k-1, k])\},$$

$$\Phi_1^k(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_k}(x_0), \\ \Psi_1^k(u) & \text{otherwise.} \end{cases}$$

We apply Proposition 3.2 (with $\phi(u) = g_0^k(u)$) and we obtain that $\Phi_1^k(.) : L^1(I_k, X) \to \mathcal{D}(I_k, X)$ is lower semicontinuous. Moreover, for any $u \in L^1(I_k, X)$ one has

$$d(g_0^k(t), F(t, \widetilde{g_0^k(u)}(t)) = d(u(t), F(t, \widetilde{g_0^k(u)}(t))\chi_{I_k \setminus I_{k-1}} \leq p_0^k(u)(t) \quad a.e.(I_k), \tag{3.6}$$

where

$$p_0^k(u)(t) = |u(t)| + p(t) + L(t)|\widetilde{g_0^k(u)}(t)|.$$

Obviously, $p_0^k : L^1(I_k, X) \to L^1(I_k, X)$ is continuous. For $m \geq 0$ set

$$p_{m+1}^k(u) = M^{m+1} \int_0^t p_0^k(u)(s) \frac{(m(t) - m(s))^m}{m!} ds + M^m \frac{(m(t))^m}{m!} \varepsilon_{m+1}.$$

and by the continuity of $p_0^k(.)$ we infer that $p_m^k : L^1(I_k, X) \to L^1(I_k, X)$ is continuous.

We shall prove, next, that for any $m \geq 1$ there exists a continuous map $g_m^k : L^1(I_k, X) \to L^1(I_k, X)$ such that

$$(a_k) \qquad g_m^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t) \quad \forall t \in I_{k-1},$$

$$(b_k) \qquad g_m^k(u) = u \quad \forall u \in \mathcal{T}_{I_k}(x_0),$$

$$(c_k) \qquad g_m^k(u)(t) \in F(t, \widetilde{g_{m-1}^k(u)}(t)) \quad a.e. \; (I_k),$$

$$(d_k) \qquad |g_1^k(u)(t) - g_0^k(u)(t)| \leq p_0^k(u)(t) + \varepsilon_0 \quad a.e. \; (I_k),$$

$$(e_k) \qquad |g_m^k(u)(t) - g_{m-1}^k(u)(t)| \leq L(t)p_{m-1}^k(u)(t) \quad a.e. \; (I_k), \quad m \geq 2.$$

Define

$$H_1^k(u) = cl\{v \in \Phi_1^k(u); \quad |v(t) - g_0^k(u)(t)| < p_0^k(u)(t) + \varepsilon_0 \quad a.e. \; (I_k)\}.$$

From (3.6), $H_1^k(u) \neq \emptyset \ \ \forall u \in L^1(I_1, X)$. Using the continuity of g_0^k, p_0^k and Lemma 2.2, we obtain a continuous selection g_1^k of H_1^k that satisfies (a_k)-(d_k).

Assume we have constructed $g_i^k(.),\ i = 1, \dots m$ satisfying (a_k)-(e_k). Then from (e_k) we have

$$\begin{aligned}&d(g_m^k(u)(t), F(t, \widetilde{g_m^k(u)}(t)) \le L(t)(|\widetilde{g_{m-1}^k(u)}(t) - \widetilde{g_m^k(u)}(t)| \le \\ &L(t)\int_0^T ML(s)p_m^k(u)(s)ds = L(t)(p_{m+1}^k(u)(t) - r_m^k(t)) < L(t)p_{m+1}^k(u)(t),\end{aligned} \tag{3.7}$$

where $r_m^k(t) := M^m \frac{(m(t))^m}{m!}(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $u \in L^1(I_k, X)$, we define

$$\begin{aligned}&\Psi_{m+1}^k(u) = \{w \in L^1(I_k, X);\ w(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}}(t) + v(t)\chi_{I_k \setminus I_{k-1}}(t),\\ &v(t) \in F(t, \widetilde{g_m^k(u)}(t)) \quad a.e.\ ([k-1, k])\},\end{aligned}$$

$$\Phi_{m+1}^k(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_k}(x_0), \\ \Psi_{m+1}^k(u) & \text{otherwise.} \end{cases}$$

With Proposition 3.2 we infer that $\Phi_{m+1}^k(.) : L^1(I_k, X) \to \mathcal{P}(L^1(I_k, X))$ is lower semicontinuous with closed decomposable and nonempty values. By (3.7) the set

$$H_{m+1}^k(u) = cl\{v \in \Phi_{m+1}^k(u);\ |v(t) - g_{m+1}^k(u)(t)| < L(t)p_{m+1}^k(u)(t)\ a.e.\ (I_k)\}$$

is nonempty for any $u \in L^1(I_k, X)$. So, applying Lemma 2.2, we deduce a continuous selection g_{m+1}^k of H_{m+1}^k, satisfying (a_k)-(e_k).

By (e_k) one has

$$|g_{m+1}^k(u) - g_m^k(u)|_{1,k} \le \frac{(Mm(k))^m}{m!}(M|p_0^k(u)|_{1,1} + \varepsilon].$$

Therefore, with a similar proof as in the case $k = 1$, we find that the sequence $\{g_m^k(u)\}_{m \in \mathbf{N}}$ converges to some $g^k(u) \in L^1(I_k, X)$ and the map $g^k(.) : L^1(I_k, X) \to L^1(I_k, X)$ is continuous.

By (a_k) we have that

$$g^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t) \quad \forall t \in I_{k-1},$$

by (b_k) $g^k(u) = u, \quad \forall u \in \mathcal{T}_{I_k}(x_0)$ and from (c_k) and the fact that F has closed values we obtain that

$$g^k(u)(t) \in F(t, \widetilde{g^k(u)}(t)), \quad a.e.\ (I_k) \quad \forall u \in L^1(I_k, X).$$

Therefore $g^k(.)$ satisfies the properties (I), (II) and (III).

Next we consider the second-order semilinear differential inclusion (1.2).

Theorem 3.4. *Consider A the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t \in \mathbf{R}}$ on the real separable Banach space X, assume that Hypothesis 3.1 is satisfied, let $x_0, x_1 \in X$ and let $\mathcal{T}^2(x_0, x_1)$ be the selection set defined in (2.13).*

Then there exists a continuous mapping $G : L^1_{loc}([0, \infty), X) \to L^1_{loc}([0, \infty), X)$ such that

(i) $G(u) \in \mathcal{T}^2(x_0, x_1), \quad \forall u \in L^1_{loc}([0, \infty), X)$,

(ii) $G(u) = u, \quad \forall u \in \mathcal{T}^2(x_0, x_1)$.

The proof of Theorem 3.4 is similar to the one of Theorem 3.3.

Remark 3.5. We recall that if Y is a Hausdorff topological space, a subspace X of Y is called retract of Y if there is a continuous map $h : Y \to X$ such that $h(x) = x, \ \forall x \in X$.

Therefore, by Theorem 3.3, for any $x_0 \in X$, the set $\mathcal{T}^1(x_0)$ of selections that correspond to solutions of (1.1) is a retract of the Banach space $L^1_{loc}([0,\infty), X)$ and by Theorem 3.4 for any $x_0, x_1 \in X$, the set $\mathcal{T}^2(x_0, x_1)$ of selections that correspond to solutions of (1.2) is a retract of $L^1_{loc}([0,\infty), X)$.

References

[1] M. Benchohra, *On second-order differential inclusions in Banach spaces*, New Zealand J. Math., **30**(2001), 5-13.

[2] M. Benchohra, S.K. Ntouyas, *Existence of mild solutions of second-order initial value problems for differential inclusions with nonlocal conditions*, Atti. Sem. Mat. Fis. Univ. Modena, **49**(2001), 351-361.

[3] A. Bressan, G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math., **90**(1988), 69-86.

[4] A. Bressan, A. Cellina, A. Fryszkowski, *A class of absolute retracts in space of integrable functions*, Proc. Amer. Math. Soc., **112**(1991), 413-418.

[5] A. Cernea, *Some qualitative properties of the solution set of infinite horizon operational differential inclusions*, Revue Roumaine Math. Pures Appl., **43**(1998), 317-328.

[6] A. Cernea, *On the relaxation theorem for semilinear differential inclusions in Banach spaces*, Pure Math. Appl., **13**(2002), 441-445.

[7] A. Cernea, *On a second-order differential inclusion*, Atti. Sem. Mat. Fis. Univ. Modena, **55**(2007), 3-12.

[8] A. Cernea, *On the solution set of some classes of nonconvex nonclosed differential inclusions*, Portugaliae Math., **65**(2008), 485-496.

[9] A. Cernea, *Lipschitz-continuity of the solution map of some nonconvex second-order differential inclusions*, Fasc. Math., **41**(2009), 45-54.

[10] F.S. De Blasi, G. Pianigiani, *Topological properties of nonconvex differential inclusions*, Nonlinear Anal., **20**(1993), 871-894.

[11] F.S. De Blasi, G. Pianigiani, V. Staicu, *Topological properties of nonconvex differential inclusions of evolution type*, Nonlinear Anal., **24**(1995), 711-720.

[12] O. Fattorini, *Second-order linear differential equations in Banach spaces*, Mathematical Studies vol. 108, North Holland, Amsterdam, 1985.

[13] H. Frankowska, *A priori estimates for operational differential inclusions*, J. Diff. Eqs., **84**(1990), 100-128.

[14] A. Fryszkowski, *Fixed Point Theory for Decomposable Sets*, Springer, Berlin, 2004.

[15] V. Staicu, *Continuous selections of solutions sets to evolutions equations*, Proc. Amer. Math. Soc., **113**(1991), 403-414.

[16] V. Staicu, *On the solution sets to differential inclusions on unbounded interval*, Proc. Edinburgh Math. Soc., **43**(2000), 475-484.

[17] V. Staicu, *Arcwise conectedness of solution sets to differential inclusions*, J. Math. Sciences, **120**(2004), 1006-1015.

[18] V. Staicu, H. Wu, *Arcwise connectedness of solution sets to lipschitzean differential inclusions*, Boll . Un. Mat. Ital. A, **5**(1991), 253-256.

[19] C. C. Travis, G. F. Webb, *Cosine families and abstract nonlinear second-order differential equations*, Acta Math. Hungarica, **32**(1978), 75-96.

Received: October 13, 2011 Accepted: January 19, 2012.