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EXISTENCE OF SOLUTIONS FOR SECOND ORDER IMPULSIVE CONTROL PROBLEMS WITH BOUNDARY CONDITIONS

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Abstract. Control of impulsive differential equations appear naturally in physical phenomena. Most often these phenomena take place during a finite time interval. This leads to the study of boundary value problems for control of impulsive differential equations. In this paper we address the problem of existence of solutions of control of impulsive differential equations of second order subjected to two-point boundary conditions. Our approach is based on the Granas topological transversality theorem and the Schauder fixed point theorem. The uniqueness of solutions is also discussed.
Key Words and Phrases: second order impulsive control problem, boundary value problems, Granas topological transversality theorem, Schauder fixed point theorem.
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1. Introduction

Many processes in the applied sciences are described by differential equations having smooth solutions. However, the situation is quite different in many physical phenomena that are subject to sudden changes in their states, such as mechanical systems with impact, biological systems (for instance heart beats, blood flows), population dynamics, theoretical physics, radiophysics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology, and so on. The appropriate mathematical models of such processes are systems of differential equations with impulses (see [2], [7], [12]). The theory of impulsive differential equations was introduced by V. D. Milman and A. D. Mishkis (see [9]). It was pointed out in [14] that impulsive control systems can be classified into three types depending on the control vectors. The control input in the first type takes place at the sudden change of some state variables. In the second type there are two kinds of control inputs continuous control input, which works on all the state variables, and impulsive control input, which works at the sudden change. Finally, in the third type the system is an impulsive system and the control is continuous.

In this paper we consider the following second order impulsive control system

$$\begin{cases} \ddot{x}\left(t\right)=F(t,x\left(t\right),\dot{x}\left(t\right)),\ t\in[0,T]-\{t_1,t_2,...,t_m\}, \\ \Delta x(t_k)=U_k(x(t_k)),\quad k=1,2,...,m, \\ \Delta\dot{x}(t_k)=V_k(\dot{x}(t_k)),\ k=1,2,...,m, \\ x(0)=x(T)=0, \end{cases} \tag{1.1}$$

where $x\in\mathbb{R}^n$ is the state variable; $F:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a piecewise continuous function; $x(t_k^+)$ and $x(t_k^-)$ represent the right limit and left limit, respectively, of the state at $t=t_k$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-), k=1,2,...,m$. The numbers t_k are called instants (or moments) of impulse and satisfy $0<t_1<t_2<...<t_m<T$. U_k and V_k are impulsive controls and represent the jumps of the state at each t_k. Notice that the moments of impulse may be fixed or may depend on the state of the system. In our study we will be concerned with fixed moments only. We refer the reader interested in the study of second order impulsive boundary value problems to the following papers and the references therein ([1], [3], [5], [6], [8], [11], [13]).

The paper is organized as follows. Section 2 is devoted to the study of the linear system corresponding to (1.1). In section 3 we state and prove the main results about the existence of solutions for (1.1).

2. Linear Problem

In this section we introduce some definitions and notations that will be used in the remainder of the paper. Then, we study the linear problem, which plays an important role in our investigation.

Let J denote the real interval $[0,T]$. For $i=1,2,...m$, consider the points $t_1,t_2,t_3,...,t_m$ such that $0<t_1<t_2<...<t_m<T$. If $I=\{t_i:i=1,2,...,m\}$ let $J'=J-I$. $PC(J)$ denotes the space of all functions $x:J\to\mathbb{R}^n$ continuous on J', and for $i=1,2,...,m$, $x(t_i^+)=\lim\limits_{\epsilon\to0+}x(t_i+\epsilon)$ and $x(t_i^-)=\lim\limits_{\epsilon\to0}x(t_i-\epsilon)$ exist, and $x(t_i^-)=x(t_i)$. This is a Banach space when equipped with the sup-norm, i.e. $\|x\|_0=\sup\limits_{t\in J}\|x(t)\|$ where $\|x(\cdot)\|$ is any norm in $\mathbb{R}^n$. Similarly, $PC^1(J)$ is the space of all functions $x\in PC(J)$, x is continuously differentiable on J', and for $i=1,2,...,m$, $\dot{x}(t_i^+)$ and $\dot{x}(t_i^-)$ exist and $\dot{x}(t_i)=\dot{x}(t_i^-)$. For $x\in PC^1(J)$ we define its norm by $\|x\|_1=\|x\|_0+\|\dot{x}\|_0$. Then $\left(PC^1(I),\|\cdot\|_1\right)$ is a Banach space.

We now consider the corresponding linear problem to (1.1).

$$\begin{cases} \ddot{x}\left(t\right)=f(t),\ t\neq t_k,t\in[0,T], \\ \Delta x(t_k)=U_k(t_k),\ k=1,2,...,m, \\ \Delta\dot{x}(t_k)=V_k(t_k),\ k=1,2,...,m, \\ x(0)=x(T)=0, \end{cases} \tag{2.1}$$

where $x\in\mathbb{R}^n$ is the state variable; U_k and V_k are impulsive controls and for every k; $k=1,2,...,m$; $0<t_1<t_2<...<t_k<...<T$.

In order to study (2.1), we first consider the problem without impulses

$$\begin{cases} \ddot{x}(t) = f(t), & t \in [0,T], \\ x(0) = x(T) = & 0 \end{cases} \tag{2.2}$$

It is well known that any solution of (2.2) is given by

$$x(t) = \int_0^T G(t,s)f(s)ds,$$

where $G(.,.) : [0,T]^2 \to \mathbb{R}$ is the Green's function and is given by

$$G(t,s) = \begin{cases} \frac{s(t-T)}{T}, 0 \le s < t \le T; \\ \frac{t(s-T)}{T}, 0 \le t \le s \le T. \end{cases} \tag{2.3}$$

Remark 2.1. *The Green's function $G(t,s)$ and its derivatives have the following properties*

(i) $0 \le |G(t,s)| \le \frac{T}{4}$.

(ii) $\displaystyle\int_0^T |G(t,s)|\, ds \le \frac{T^2}{8}$.

(iii) $\displaystyle\int_0^T \left|\frac{\partial G}{\partial t}(t,s)\right| ds \le \frac{T}{2}$.

(iv) $\left|\frac{\partial G}{\partial t}(t,s)\right| \le 1$ *and* $\left|\frac{\partial G}{\partial s}(t,s)\right| \le 1$

(v) $\left|\frac{\partial^2 G}{\partial s \partial t}\right| = \frac{1}{T}$.

Lemma 2.2. *The solution of problem (2.1) is given by*

$$x(t) = \int_0^T G(t,s)f(s)ds - \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial s} U_k(t_k) + \sum_{k=1}^{m} G(t,t_k)V_k(t_k). \tag{2.4}$$

Proof. We shall use the superposition principle, and write $x(t) = y(t) + z(t) + w(t)$, where $y(t)$ solves the problem

$$\begin{cases} \ddot{y}(t) = f(t), & t \in J', \\ \Delta y(t_k) = 0, \ k = 1,2,...,m, \\ \Delta \dot{y}(t_k) = 0, \ k = 1,2,...,m, \\ y(0) = y(T) = 0, \end{cases} \tag{2.5}$$

$z(t)$ solves the problem

$$\begin{cases} \ddot{z}(t) = 0 \ , \quad t \in J', \\ \Delta z(t_k) = U_k(t_k), \ k = 1,2,...,m, \\ \Delta \dot{z}(t_k) = 0, \ k = 1,2,...,m, \\ z(0) = z(T) = 0, \end{cases} \tag{2.6}$$

and $w(t)$ solves the problem

$$\begin{cases} \ddot{w}(t) = 0 \quad , \quad t \in J', \\ \Delta w(t_k) = 0, \; k = 1, 2, ..., m, \\ \Delta \dot{w}(t_k) = V_k(t_k), \; k = 1, 2, ..., m, \\ w(0) = w(T) = 0. \end{cases} \tag{2.7}$$

Then simple computations lead to (2.4).

3. Nonlinear Problem

In this section we will present our main results on the existence of solutions for nonlinear boundary value problems for second order impulsive control systems.

Consider the problem

$$\begin{cases} \ddot{x}(t) = F(t, x(t), \dot{x}(t)), \quad t \in J', \\ \Delta x(t_k) = U_k(x(t_k)), \; k = 1, 2, ..., m, \\ \Delta \dot{x}(t_k) = V_k(\dot{x}(t_k)), \; k = 1, 2, ..., m, \\ x(0) = x(T) = 0, \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state variable; $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise continuous function; U_k and V_k are impulsive controls with $0 < t_1 < t_2 < ... < t_m < T$.

It follows from Lemma (2) that any solution of (3.1) satisfies

$$\begin{aligned} x(t) = \int_0^T G(t,s)F(s, x(s), \dot{x}(s))ds - \sum_{k=1}^{m} \frac{\partial G(t, t_k)}{\partial s} U_k(x(t_k)) \\ + \sum_{k=1}^{m} G(t, t_k)V_k(\dot{x}(t_k)). \end{aligned} \tag{3.2}$$

Theorem 3.1. *Assume that the following conditions hold*

(H1) $F(\cdot, x, y)$ is continuous on J' and $F(t, \cdot, \cdot)$ satisfies a Lipschitz condition

$$\|F(t, x_1, y_1) - F(t, x_2, y_2)\| \leq \alpha \|x_1 - y_1\| + \beta \|x_2 - y_2\|.$$

(H2) U_k, V_k are Lipschitz continuous with Lipschitz constant l_k and p_k, $k = 1, 2, ..., m$, respectively, with

$$\delta := \max\{(1 + \frac{1}{T})\sum_{k=1}^{m} l_k, \; (\frac{T}{4} + 1)\sum_{k=1}^{m} p_k\} < 1,$$

(H3) $\gamma T(T + 4) < 8(1 - \delta)$, where $\gamma = \max\{\alpha, \beta\}$.

Then problem (3.1) has a unique solution.

Proof. Define an operator $\varphi : PC^1(J) \to PC^1(J)$ by

$$(\varphi x)(t) = \int_0^T G(t,s)F(s,x(s),\dot{x}(s))ds - \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial s} U_k(x(t_k)) + \sum_{k=1}^{m} G(t,t_k)V_k(\dot{x}(t_k)).$$

It is clear that any solution of (3.1) is a fixed point of φ and vice-versa. We shall show that φ is a contraction. Let $x, y \in PC^1(J)$, then

$$\begin{aligned}\|\varphi(x(t)) - \varphi(y(t))\| \le & \int_0^T |G(t,s)| \, \|F(s,x(s),\dot{x}(s)) - F(s,y(s),\dot{y}(s))\| \, ds \\ & + \sum_{k=1}^{m} \left| \frac{\partial G(t,t_k)}{\partial s} \right| \|U_k(x(t_k)) - U_k(y(t_k))\| \\ & + \sum_{k=1}^{m} |G(t,t_k)| \, \|V_k(\dot{x}(t_k)) - V_k(\dot{y}(t_k))\| \\ \le & \int_0^T |G(t,s)| \left(\alpha \|x(s) - y(s)\| + \beta \|\dot{x}(s) - \dot{y}(s)\|\right) ds \\ & + \sum_{k=1}^{m} \left| \frac{\partial G(t,t_k)}{\partial s} \right| l_k \|x(t_k) - y(t_k)\| \\ & + \sum_{k=1}^{m} |G(t,t_k)| \, p_k \|\dot{x}(t_k) - \dot{y}(t_k)\| .\end{aligned}$$

Now, using conditions *(H1)*, *(H2)* and the above remark we get

$$\|\varphi(x) - \varphi(y)\|_0 \le \gamma \frac{T^2}{8} \|x - y\|_1 + \sum_{k=1}^{m} l_k \|x - y\|_0 + \frac{T}{4} \sum_{k=1}^{m} p_k \|\dot{x} - \dot{y}\|_0 . \tag{3.3}$$

Next, we have that

$$\frac{d}{dt}\varphi(x)(t) = \int_0^T G_t(t,s)F(s,x(s),\dot{x}(s))ds - \sum_{k=1}^{m} \frac{\partial^2 G(t,t_k)}{\partial t \partial s} U_k(x(t_k)) + \sum_{k=1}^{m} G_t(t,t_k)V_k(\dot{x}(t_k)).$$

The above inequality implies

$$\begin{aligned}\left\| \frac{d}{dt}\varphi(x) - \frac{d}{dt}\varphi(y) \right\|_0 &\leq \sup_{t\in J}\{ \int_0^T |G_t(t,s)| \, [\alpha \|x-y\|_0 + \beta \|\dot{x}-\dot{y}\|_0] ds \\ &+ \sum_{k=1}^m \left| \frac{\partial^2 G(t,t_k)}{\partial t \partial s} \right| \|U_k(x) - U_k(y)\|_0 \\ &+ \sum_{k=1}^m |G_t(t,t_k)| \, \|V_k(\dot{x}) - V_k(\dot{y})\|_0 \}.\end{aligned}$$

Hence

$$\left\| \frac{d}{dt}\varphi x - \frac{d}{dt}\varphi y \right\|_0 \leq \gamma \frac{T}{2} \|x-y\|_1 + \sum_{k=1}^m \frac{1}{T} l_k \|x-y\|_0 + \sum_{k=1}^m p_k \|\dot{x}-\dot{y}\|_0 . \qquad (3.4)$$

From (3.3) and (3.4) we get

$$\begin{aligned}\|\varphi x - \varphi y\|_1 &= \|\varphi(x) - \varphi(y)\|_0 + \left\| \frac{d}{dt}\varphi(x) - \frac{d}{dt}\varphi(y) \right\|_0 \\ &\leq \gamma(\frac{T^2}{8} + \frac{T}{2}) \|x-y\|_1 + (1+\frac{1}{T}) \sum_{k=1}^m l_k \|x-y\|_0 \\ &+ (\frac{T}{4}+1) \sum_{k=1}^m p_k \|\dot{x}-\dot{y}\|_0 .\end{aligned}$$

Letting $\delta = \max\{(1+\frac{1}{T})\sum_{k=1}^m l_k, \, (\frac{T}{4}+1)\sum_{k=1}^m p_k\}$ we get

$$\|\varphi x - \varphi y\|_1 \leq (\gamma(\frac{T^2}{8} + \frac{T}{2}) + \delta) \|x-y\|_1 .$$

It follows from condition *(H3)* that φ is contraction. By the Banach fixed point theorem φ has a unique fixed point x, which is the unique solution of (3.1).

Example 3.2. *Consider the impulsive control system*

$$\begin{cases} \ddot{x}(t) = \frac{1}{2}\cos x(t), \ t \neq t_k, t \in J, \\ \Delta x(t_k) = \frac{5}{32k} x(t_k), \ k = 1,2, \\ \Delta \dot{x}(t_k) = \frac{k}{8} \dot{x}(t_k), \ k = 1,2, \\ x(0) = x(1) = 0, \end{cases} \qquad (3.5)$$

where $J = [0,1]$, $t_1 = \frac{1}{2}$, $t_2 = \frac{3}{4}$. We see that $\gamma = \frac{1}{2}$ and $\delta = \frac{15}{32}$. So, for (3.5) we conclude that there is a unique solution. While, if we take $\Delta x(t_k) = \frac{1}{k+1} x(t_k)$ and $\Delta \dot{x}(t_k) = \frac{1}{2k+1} \dot{x}(t_k)$ we find $\gamma = \frac{1}{2}$ and $\delta = \frac{5}{3}$; so we cannot conclude any thing about uniqueness since (H3) is not satisfied.

Theorem 3.3. *Suppose the following conditions hold*

(H4) $F(\cdot,x,y): J \to \mathbb{R}^n$ *is continuous on* J'*, there exists* $h: J\times\mathbb{R}_+ \to \mathbb{R}_+$*, a Caratheodory function, nondecreasing with respect to its second argument such that*

$$\|F(t,x,y)\| \le h(t, \|x\|+\|y\|), \quad \text{for all } x,y\in\mathbb{R}^n \text{ and a.e. } t\in[0,T].$$

(H5) $U_k:\mathbb{R}^n\to\mathbb{R}^n$ *is continuous and there exists* $l_k>0$ *such that*

$$\|U_k(x)\|_0 \le l_k, k=1,2,...,m.$$

(H6) $V_k:\mathbb{R}^n\to\mathbb{R}^n$ *is continuous and there exists* $p_k>0$ *such that*

$$\|V_k(x)\|_0 \le p_k, \quad k=1,2,...,m.$$

(H7) $\lim\limits_{\rho\to+\infty}\sup \frac{1}{\rho}\left(\int_0^T h(t,\rho)dt + \sum_{k=1}^m \left(\left(\frac{4(T+1)}{T(T+4)}\right)l_k + p_k\right)\right) < \frac{4}{T+4}$

Then System (3.1) has at least one solution.

Proof. The proof is given in two steps.

Step 1. A priori bound on solutions. Recall that solutions of (3.1) satisfy

$$x(t) = \int_0^T G(t,s)F(s,x(s),\dot{x}(s))ds - \sum_{k=1}^m \frac{\partial G(t,t_k)}{\partial s}U_k(x(t_k)) + \sum_{k=1}^m G(t,t_k)V_k(\dot{x}(t_k)),$$

and

$$\dot{x}(t) = \int_0^T G_t(t,s)F(s,x(s),\dot{x}(s))ds - \sum_{k=1}^m \frac{\partial^2 G(t,t_k)}{\partial t\partial s}U_k(x(t_k)) + \sum_{k=1}^m G_t(t,t_k)V_k(\dot{x}(t_k)).$$

It is easy to see that

$$\|x(t)\| \le \frac{T}{4}\int_0^T \|F(s,x(s),\dot{x}(s))\|\,ds + \sum_{k=1}^m \|U_k(x(t_k))\| + \frac{T}{4}\sum_{k=1}^m \|V_k(\dot{x}(t_k))\|,$$

and

$$\|\dot{x}(t)\| \le \int_0^T \|F(s,x(s),\dot{x}(s))\|\,ds + \frac{1}{T}\sum_{k=1}^m \|U_k(x(t_k))\| + \sum_{k=1}^m \|V_k(\dot{x}(t_k))\|.$$

Conditions *(H4), (H5) and (H6)* lead to

$$\|x\|_0 + \|\dot{x}\|_0 \le (\frac{T}{4}+1)\int_0^T h(s, \|x\|_0+\|\dot{x}\|_0)ds + \sum_{k=1}^m((1+\frac{1}{T})l_k + (\frac{T}{4}+1)p_k).$$

Since $\|x\|_1 = \|x\|_0 + \|\dot{x}\|_0$ and h is nondecreasing, then

$$\|x\|_1 \le (\frac{T}{4}+1)\int_0^T h(s,\|x\|_1)ds + \sum_{k=1}^m((1+\frac{1}{T})l_k + (\frac{T}{4}+1)p_k).$$

Let

$$\rho_0 = \|x\|_1\,.$$

Then the above inequality gives

$$\frac{4}{(T+4)} \leq \frac{1}{\rho_0}\left(\int_0^T h(s,\rho_0)ds + \sum_{k=1}^{m}\left(\left(\frac{4(T+1)}{T(T+4)}\right)l_k + p_k\right)\right). \tag{3.6}$$

Condition *(H7)* implies that there exists $r > 0$ such that for all $\rho > r$ we have

$$\frac{1}{\rho}\left(\int_0^T h(s,\rho)ds + \sum_{k=1}^{m}\left(\left(\frac{4(T+1)}{T(T+4)}\right)l_k + p_k\right)\right) < \frac{4}{(T+4)}. \tag{3.7}$$

Comparing (3.6) and (3.7) we can see that $\rho_0 \leq r$. Hence, all possible solutions of (3.1) satisfy

$$\|x\|_1 \leq r\ .$$

Step 2. Existence of solutions. Let $\Omega = \{x \in PC^1(J) : \|x\|_1 < r+1\}$. Then Ω is an open convex subset of $PC^1(J)$.
Define an operator $H : [0,1] \times \Omega \to PC^1(J)$ by

$$H(\lambda, x)(t) = \lambda \int_0^T G(t,s)F(s,x(s),\dot{x}(s))ds - \lambda \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial t} U_k(x(t_k))$$
$$+ \lambda \sum_{k=1}^{m} G(t,t_k)V_k(\dot{x}(t_k)), \qquad 0 \leq \lambda \leq 1.$$

$H(\lambda,\cdot) : \bar{\Omega} \to PC^1(J)$ is compact since it is the sum of two operators, the first one is a compact integral operator with kernel the Green's function, and the second is a finite rank operator, which is also compact (see [10], [6]). Also, it follows from the previous step that $H(\lambda.\cdot)$ has no fixed point on $\partial\Omega$, the boundary of Ω. Consequently, see [4], $H(\lambda,\cdot)$ is an admissible homotopy between the constant map $H(0,\cdot) \equiv 0$ and $H(1,\cdot) \equiv \varphi$. Since $H(0,\cdot)$ is essential then $H(1,\cdot)$ is essential, which implies that $\varphi \equiv H(1,\cdot)$ has a fixed point in Ω. This fixed point is a solution of our problem.

Theorem 3.4. *Suppose the following conditions hold*

(H8) there exists $h \in L^1(J)$ such that

$$\|F(t,x,y)\| \leq h(t) \ \ \textit{for all } x,y \in \mathbb{R}^n \textit{ and a.e. } t \in [0,T]..$$

(H9) $U_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists $\alpha_k > 0$ such that

$$\|U_k(x)\|_0 \leq \alpha_k \|x\|_0\,, k = 1,2,...,m.$$

(H10) $V_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists $\beta_k > 0$ such that

$$\|V_k(y)\|_0 \leq \beta_k \|y\|_0\,, \quad k = 1,2,...,m.$$

(H11) $\mu < 1$, where $\mu = \max\left\{\left(1+\frac{1}{T}\right)\sum_{k=1}^{m}\alpha_k, (\frac{T}{4}+1)\sum_{k=1}^{m}\beta_k\right\}$.

Then system (3.1) has at least one solution.

Proof. We proceed as in step 1 of the proof of the previous result to obtain a priori bound on solutions. It is clear that

$$\|x(t)\| \leq \frac{T}{4}\int_0^T \|F(s,x(s),\dot{x}(s))\|\,ds + \sum_{k=1}^{m}\|U_k(x(t_k))\| + \frac{T}{4}\sum_{k=1}^{m}\|V_k(\dot{x}(t_k))\|\,,$$

and

$$\|\dot{x}(t)\| \leq \int_0^T \|F(s,x(s),\dot{x}(s))\| \, ds + \frac{1}{T}\sum_{k=1}^{m} \|U_k(x(t_k))\| + \sum_{k=1}^{m} \|V_k(\dot{x}(t_k))\| \, .$$

(H8), (H9) and *(H10)* imply

$$\|x\|_0 + \|\dot{x}\|_0 \leq (\frac{T}{4}+1) \, \|h\|_{L^1} + \sum_{k=1}^{m}\left(\left(1+\frac{1}{T}\right)\alpha_k \, \|x\|_0 + \left(\frac{T}{4}+1\right)\beta_k \, \|\dot{x}\|_0\right).$$

Letting $\mu = \max\left\{\left(1+\frac{1}{T}\right)\sum_{k=1}^{m}\alpha_k, (\frac{T}{4}+1)\sum_{k=1}^{m}\beta_k\right\}$ we get

$$\|x\|_1 \leq (\frac{T}{4}+1) \, \|h\|_{L^1} + \mu \, \|x\|_1 \, .$$

Then

$$(1-\mu) \, \|x\|_1 \leq (\frac{T+4}{4}) \, \|h\|_{L^1} \, .$$

Condition *(H11)* gives

$$\|x\|_1 \leq \frac{(T+4)}{4(1-\mu)} \, \|h\|_{L^1} \, .$$

Step 2. Existence of solutions. Define a nonlinear operator $\psi : PC^1(J) \to X_0$ where $X_0 := \{x \in PC^1(J) : x(0) = x(T) = 0\}$ by

$$(\psi \, (x))(t) = \int_0^T G(t,s)F(s,x(s),\dot{x}(s))ds - \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial s} U_k(x(t_k)) + \sum_{k=1}^{m} G(t,t_k)V_k(\dot{x}(t_k)).$$

Let $D := \{x \in X_0 : \|x\|_1 \leq \frac{(T+4)}{4(1-\mu)} \|h\|_{L^1}\}$. We can prove that ψ is continuous, maps the closed convex set D into itself and $\overline{\psi(D)}$ is compact. By the Schauder fixed point theorem, we conclude that ψ has a fixed point in D, which is a solution of our problem (3.1).

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