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MULTIVALUED FIXED POINT THEOREMS WITHOUT STRONG COMPACTNESS VIA A GENERALIZATION OF MIDPOINT CONVEXITY

IRENE BENEDETTI*, STELLA BOLOGNINI AND ANNA MARTELLOTTI

Dept. of Mathematics and Computer Science, University of Perugia, I-06123 Italy
E-mail: irene.benedetti@dmi.unipg.it, stellabolognini@googlemail.com, amart@dipmat.unipg.it

Abstract. We investigate a generalization of midpoint convexity for multivalued maps, and derive fixed points theorems under this more general assumption without requiring any strong compactness condition. As an application we prove the existence of social equilibria for games with n players.
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1. Introduction

Let X a Banach space, and let X_w denote the space X endowed with the weak topology. In this paper we shall deal with fixed point theorems for upper semicontinuous multimaps from X to X_w that map a weakly compact convex set into itself. The classical Brower's Fixed Point Theorem has been extended to multivalued maps (multimaps) by Kakutani in 1941 ([13]). As it is well known, in this pioneering work Kakutani proves the existence of at least a fixed point for an upper semicontinuous multimap that maps a closed convex bounded set K into itself in finite dimensional Banach spaces. In the framework of more general Banach spaces, Bohnenblust-Karlin in 1950 extend the Schauder's fixed point theorem to multimaps ([4]).
If the set K is compact and convex, Kakutani's result has been generalized to locally convex topological vector spaces (LCTVS), similarly to the classical Tychonoff's extension of Schauder's Theorem; this was done e.g. in Gliksberg [11] and Fan [10]. One of the main improvement, when dealing with LCTVS, lies in the possibility to assume weak compactness of the set K instead of the strong one. The drawback however, is that one has to deal with the regularity of the multimap with respect to the weak topology. A more relaxed assumption is the upper semicontinuity from X to X_w. On the other hand, an upper semicontinuous multimap from X to X_w that maps a weakly compact set K into itself may not have a fixed point, as shown, for instance, by the classical example of Kakutani (1941) of a fixed point free map on the unit ball of ℓ^2.

*Corresponding author.

In order to obtain a fixed point theorem for an upper semicontinuous multimap F from X to X_w we propose an extension of the concept of midpoint linearity introduced in [8]. The idea came from the very proof of Ky Fan's Theorem in [10], where the convexity of the fixed point set for the multimap $G(x) = F(x) + V$, (where V is any weak neighborhood of the origin) is needed.
Our extension of midpoint linearity is twofold: first we provide a suitable multivalued version, and we also extend the original definition (for both single-valued and multivalued maps) to the weak topology of X.
Actually, to state our fixed point theorem we need something weaker than midpoint linearity; more precisely it is enough to require a form of piecewise midpoint linearity. Indeed, we obtain two different fixed point theorems: a fixed point theorem for α-nonexpansive strongly midpoint linear multimaps (Theorem 4.2) which is an extension to the multivalued case and to general Banach spaces of Theorem 4.5 in [8] and which is based on the classical generalization to multimaps of Darbo's fixed point Theorem (see e.g. Corollary 3.3.1. in [12]). Then we obtain a second fixed point theorem for weakly midpoint linear multimaps, which are assumed to be simply X to X_w upper semicontinuous (see Theorem 4.3).
The midpoint linearity is a generalization of the convexity for multimaps introduced by Nikodem in [15]. More precisely in Corollary 3.1 we prove that a closed-valued midpoint convex multimap, which is upper semicontinuous from X to X_w is both strongly and weakly midpoint linear.
Finally, this last result, together with the "weak" fixed point theorem (Theorem 4.3), find an application in the last section, where we prove the existence of a Debreu's social equilibrium for a game with n players. The idea of deducing the existence of social equilibria from fixed point theorems is widely spread in the literature. Several authors have given contributions in this direction after Debreu's initial result. See for instance [16] and [1] where social equilibria are obtained under different assumptions.

2. Midpoint linearity

In the sequel we recall some definitions and results from multivalued analysis which will be used later. For what concerns standard definitions about maps and multimaps we refer to [2].
In the whole paper $(X, \|\cdot\|)$ is a Banach space, X^* its dual space. X_w denotes the space X endowed with the weak topology; X_1 and X_1^* denote the closed unit ball of X and of X^* respectively, $K \subseteq X$ denotes a convex subset of X and $F : K \multimap K$ will be a proper multimap with convex values. Finally we denote with $\mathcal{B}$ a fundamental system of neighborhoods of the null element 0 in X with respect to the weak topology of X, defined by

$$V = V(x_1^*, \dots, x_n^*, \varepsilon) = \{x \in X \ : \ |x_i^*(x)| < \varepsilon, \, i = 1, \dots, n\}$$

and

$$\overline{V} = \{x \in X \ : \ |x_i^*(x)| \leq \varepsilon, \, i = 1, \dots, n\}.$$

Definition 2.1. A map $f : K \to \mathbb{R}$ is said to be *quasiconvex* if the set

$$L_\alpha^- = \{x \in K \ : \ f(x) \leq \alpha\} \tag{2.1}$$

is convex.

Let $A \subset X$, with the symbol $d(x, A)$ we denote the usual distance from a set

$$d(x, A) = \inf\{\|x - y\|;\ y \in A\}.$$

Definition 2.2. A set $A \subset X$ is said to be *proximinal* if for any $x \in X \setminus A$, it holds

$$P_A(x) = \{y \in A\ :\ \|x - y\| = d(x, A)\} \neq \emptyset.$$

In [8] for a map $f : K \to X$ the following definition has been proposed.

Definition 2.3. A continuous map $f : K \to X$ is said *midpoint linear* if for every positive number $r > 0$ the following property holds

$$f\left(\frac{x+y}{2}\right) \in \frac{x+y}{2} + rX_1,$$

whenever $x, y \in K,\ f(x) \in x + rX_1,\ f(y) \in y + rX_1$.

Observe that this is equivalent to saying that for any $r > 0$ the set

$$\Phi_r = \{x \in K\ :\ x \in f(x) + rX_1\}$$

is convex.
This suggests in the multivalued case the following two concepts.

Definition 2.4. A multimap $F : K \multimap X$ is said to be *strongly midpoint linear* if for any $r > 0$ the set

$$F_r = \{x \in K\ :\ x \in F(x) + rX_1\}$$

is convex.

Definition 2.5. A multimap $F : K \multimap X$ is said to be *weakly midpoint linear* if for any $V \in \mathcal{B}$ the set

$$F_V = \{x \in K\ :\ x \in F(x) + \overline{V}\}$$

is convex.

Remark 2.1. A single valued map $f : K \to X$ is then strongly midpoint linear if and only if the multimap $F = \{f\}$ is strongly midpoint linear in the sense of Definition 2.4.
In analogy we shall say that f is *weakly midpoint linear* provided the multimap $F = \{f\}$ is weakly midpoint linear.

In order to compare these two concepts we shall introduce the map $d : K \to \mathbb{R}$ defined as

$$d(x) = \inf_{y \in F(x)} \|x - y\| \tag{2.2}$$

and for $x^* \in X^*$ the map $d_{x^*} : K \to \mathbb{R}$ defined as

$$d_{x^*}(x) = \inf_{z \in F(x)} |x^*(z - x)|. \tag{2.3}$$

Proposition 2.1. *If a multimap $F : K \multimap X$ is strongly midpoint linear then the map $d : K \to \mathbb{R}$ defined in* (2.2) *is quasiconvex.*
Moreover if F has proximinal values the converse implication holds.

Proof. Assume F strongly midpoint linear. Let x_1, x_2 be such that $d(x_1) \leq \alpha$ and $d(x_2) \leq \alpha$. Then $\inf_{y_i \in F(x_i)} \|x_i - y_i\| \leq \alpha,\ i = 1, 2$. Hence for any $\varepsilon > 0$ there exists $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ such that $\|x_i - y_i\| < \alpha + \varepsilon$, $i = 1, 2$. Then for any $\varepsilon > 0$ $x_1, x_2 \in F_{\alpha+\varepsilon}$. Since F is strongly midpoint linear the set $F_{\alpha+\varepsilon}$ is convex and then $x_\lambda = \lambda x_1 + (1-\lambda)x_2 \in F_{\alpha+\varepsilon}$ for all $\lambda \in [0, 1]$. Then $y_\lambda \in F(x_\lambda)$ exists such that $\|x_\lambda - y_\lambda\| < \alpha + \varepsilon$, and hence $d(x_\lambda) < \alpha + \varepsilon$, for any $\varepsilon > 0$, so $d(x_\lambda) \leq \alpha$.
Assume now that d is quasiconvex. Let $\alpha > 0$ and $x_1, x_2 \in F_\alpha$. Then $d(x_i) \leq \alpha$, $i = 1, 2$ and $x_1, x_2 \in L^-_\alpha$ (with L^-_α defined in (2.1)).
From the convexity of L^-_α we have $x_\lambda = \lambda x_1 + (1-\lambda)x_2 \in L^-_\alpha$ for all $\lambda \in [0, 1]$. Hence, since F has proximinal values, there exists $y_\lambda \in F(x_\lambda)$ such that $\|x_\lambda - y_\lambda\| \leq \alpha$; then $x_\lambda \in F_\alpha$, for all $\lambda \in [0, 1]$. Therefore F is strongly midpoint linear. □

Then for single valued maps we have the following characterization.

Corollary 2.1. *A map $f : K \to X$ is strongly midpoint linear if and only if the map $d : K \to \mathbb{R}$ defined in* (2.2) *is a quasiconvex map.*

On the contrary in the multivalued case the quasiconvexity of the map d implies the midpoint linearity only with extra assumptions as the following example shows.

Example 2.1. Consider the multimap $F : \mathbb{R} \multimap \mathbb{R}$ defined as

$$F(x) = \begin{cases} [x+1, +\infty) & x \in \mathbb{R} \setminus \mathbb{Q} \\ ]x+1, +\infty) & x \in \mathbb{Q} \end{cases}.$$

We observe that, since $d(x) \equiv 1$ for any $x \in \mathbb{R}$, d is a quasiconvex map. On the other hand the multimap F is not strongly midpoint linear, indeed the set

$$F_1 = \{x \in \mathbb{R} \ : \ [x-1, x+1] \cap F(x) \neq \emptyset\} = \mathbb{R} \setminus \mathbb{Q}$$

is not a convex set.

In analogy with Proposition 2.1 for the weak midpoint linearity we have the following relation.

Proposition 2.2. *If the multimap $F : K \multimap X$ is weakly midpoint linear, then for all $x^* \in X^*$ the map $d_{x^*} : K \to \mathbb{R}$ defined in* (2.3) *is quasiconvex.*

The proof is substantially analogous to that of Proposition 2.1.
As for the strong midpoint linearity, in the single valued case the implication can be reversed.

Corollary 2.2. *A map $f : K \to X$ is weakly midpoint linear if and only if for any $x^* \in X^*$ the map d_{x^*} is quasiconvex.*

Something stricter can be said if the multimap $F : K \multimap X$ has convex values.

Proposition 2.3. *Let $F : K \multimap X$ be a weakly midpoint linear multimap with convex values, then the map d defined in* (2.2) *is quasiconvex.*

Proof. From Proposition 2.2 there follows that the map d_{x^*} is quasiconvex for all $x^* \in X^*$. Moreover

$$d(x) = \inf_{y \in F(x)} \|x - y\| = \inf_{y \in F(x)} \sup_{x^* \in X_1^*} |x^*(x - y)|.$$

Let $x \in X$ and consider the map $(y, x^*) \mapsto x^*(x - y)$, which is concave-convex and continuous with respect to $(\| \cdot \|, w^*)$ topologies. Then, when $F(x)$ is equipped with the norm topology, and X_1^* with the weak*-topology, the following minimax equality holds (see e.g. Theorem 3.1 of [17])

$$\inf_{y \in F(x)} \sup_{x^* \in X_1^*} x^*(x - y) = \sup_{x^* \in X_1^*} \inf_{y \in F(x)} x^*(x - y).$$

Moreover, easily

$$\sup_{x^* \in X_1^*} x^*(x - y) = \sup_{x^* \in X_1^*} |x^*(x - y)|.$$

So

$$\begin{aligned}\inf_{y \in F(x)} \sup_{x^* \in X_1^*} |x^*(x - y)| &= \inf_{y \in F(x)} \sup_{x^* \in X_1^*} x^*(x - y) = \sup_{x^* \in X_1^*} \inf_{y \in F(x)} x^*(x - y) \\ &\leq \sup_{x^* \in X_1^*} \inf_{y \in F(x)} |x^*(x - y)| \\ &\leq \inf_{y \in F(x)} \sup_{x^* \in X_1^*} |x^*(x - y)|,\end{aligned}$$

where the last inequality is always trivially true. Thus

$$\inf_{y \in F(x)} \sup_{x^* \in X_1^*} |x^*(x - y)| = \sup_{x^* \in X_1^*} \inf_{y \in F(x)} |x^*(x - y)|$$

Therefore

$$d(x) = \sup_{x^* \in X_1^*} d_{x^*}(x)$$

Since , according to Proposition 2.2, each d_{x^*} is quasiconvex, d is quasiconvex too. □

The following example proves that the converse implication fails to be true.

Example 2.2. Let $I = [0, 1]$, $X = \mathcal{C}(I)$ and let H be a closed non proximinal subspace of X of finite codimension with

$$H^\perp = \operatorname{span}\{x_*^1, \dots, x_*^n\},$$

with x_i^* determined by a probability measure (this is always possible see [6], [7], [18]). Let X^+ be the usual order cone of X, and $D = \{x \in X^+ \ : \ \min_{0 \leq t \leq 1} |x(t)| = 0\}$.

Fix any $x_0 \in X$ such that $P_H(x_0) = \emptyset$ and $u \in H$ with $\|u\| = 1$. Let $r = d(x_0, H)$, and define $F : X^+ \multimap X^+$ as follows

$$F(x) = \begin{cases} x + ru & \text{if} \quad x \in D \\ x - x_0 + H & \text{if} \quad x \notin D \end{cases}$$

Easily we prove that $d(x) = r$, for $x \in X^+$. Hence d is quasiconvex.
However F is not weakly midpoint linear.
In fact if x_1, $x_2 \in D$ are such that $Z(x_1) \cap Z(x_2) = \emptyset$, where

$$Z(x) = \{t \in I \ : \ x(t) = 0\},$$

then $x_\lambda = \lambda x_1 + (1-\lambda)x_2 \notin D$ for every $\lambda \in]0,1[$ (recall that $x_1 \geq 0$ and $x_2 \geq 0$).
Take any $x^* \in H^\perp$; then

$$d_{x^*}(x_i) = rx^*(u) = 0 \ \text{ (for } u \in H \text{ and } x^* \in H^\perp),$$

whence $x_i \in F_V$, where $V = V(x^*, \varepsilon)$, $\varepsilon > 0$.
On the other side

$$d_{x^*}(x_\lambda) = \inf_{\eta \in H} |-x^*(x_0) + x^*(\eta)| = |x^*(x_0)|.$$

As $x_0 \notin H$ necessarily $x_i^*(x_0) \neq 0$ for at least one index $i = \{1, \ldots, n\}$.
Let $\varepsilon < |x_i^*(x_0)|$ and $V = V(x^*, \varepsilon)$. Then $x_1, x_2 \in F_V$ while $x_\lambda \notin F_V$.

As an immediate consequence of Proposition 2.3 we have the following result.

Theorem 2.1. *Let $F : K \multimap X$ be a multimap with convex and proximinal values. Then the weak midpoint linearity of F implies the strong midpoint linearity of F.*

Consequently, for single valued maps, the weak midpoint linearity always implies the strong midpoint linearity. On the other hand, the converse may fail to be true, as the following example shows.

Example 2.3. Let $f : L^1([0,1];\mathbb{R}) \to L^1([0,1];\mathbb{R})$ be a map defined as $f(x) = \varphi(\|x\|)x$, where $\varphi : [0,\infty) \to [0,1]$,

$$\varphi(t) = \begin{cases} 1-t & t \leq 1 \\ 0 & t > 1 \end{cases}$$

The map f is strongly midpoint linear (compare with [8]).
Let $\mathbf{1}_D$ denote the indicator map of set $D \subset \mathbb{R}$ and x^* be the continuous linear functional in $L^\infty([0,1])$ determined by $2\,\mathbf{1}_{[0,\frac{1}{2}]}$.
Let $V = V(x^*, \varepsilon)$, with $0 < \varepsilon < \dfrac{1}{2}$ and consider $x = 2\varepsilon \mathbf{1}_{[0,\frac{1}{2}]}$. Then

$$\|x\|_1 = \int_0^{1/2} 2\varepsilon\, dt = \varepsilon < 1 \ \text{ and } \ |x^*(x)| = \int_0^{1/2} 4\varepsilon\, dt = 2\varepsilon.$$

Hence

$$|x^*[x - f(x)]| = |x^*[x - (1 - \|x\|)x]| = \|x\| \cdot |x^*(x)| = 2\varepsilon^2 < \varepsilon$$

then $x \in \Phi_V = \{x \in L^1([0,1];\mathbb{R}) \ : \ x \in f(x) + \overline{V}\}$.
Let now $y = 9\,\mathbf{1}_{[\frac{1}{2},1]}$ then both $x^*(y) = 0$ and $x^*[y - f(y)] = 0$ (since x^* and y have essentially disjoint supports), so $y \in \Phi_V$. Consider $\xi = \dfrac{2}{3}x + \dfrac{1}{3}y$; since both x and y are non-negative, we have

$$\|\xi\|_1 = \frac{2}{3}\|x\|_1 + \frac{1}{3}\|y\|_1 = \frac{2}{3}\varepsilon + \frac{3}{2} > 1$$

Therefore $f(\xi) = 0$; hence

$$x^*(\xi - f(\xi)) = x^*(\xi) = \frac{2}{3}x^*(x) + \frac{1}{3}x^*(y) = \frac{2}{3}\,2\varepsilon = \frac{4}{3}\varepsilon > \varepsilon.$$

Thus $x, y \in \Phi_V$, while $\xi \notin \Phi_V$.

3. Midpoint convexity versus midpoint linearity

The concept of midpoint convexity for multimaps has been introduced by Nikodem in [15] and furtherly investigated in [5].

Definition 3.1. A multimap $F : K \multimap X$ is said to be *convex* if for any $x,\, y \in K$

$$\lambda\, F(x) + (1-\lambda)\, F(y) \subseteq F\left(\lambda\, x + (1-\lambda)\, y\right). \tag{3.1}$$

If (3.1) holds only with $\lambda = \dfrac{1}{2}$, F is said to be *midpoint convex.*
In order to compare these concepts with the midpoint linearities introduced in Section 2, we shall need some preliminary results.

Proposition 3.1 ([5])**.** *Let $A \subset Y$ be a midpoint convex subset of a topological vector space Y (namely if $x, y \in A$, then $\dfrac{x+y}{2} \in A$); then $\overline{A}$ is a convex set.*

Proposition 3.2. *Let $F : K \multimap X$ be a midpoint convex multimap with closed values. Then F has convex values.*

Proof. The midpoint convexity of F immediately implies that F has midpoint convex values.
Then the assertion follows from Proposition 3.1. □

It is obvious that each convex multimap is midpoint convex. The next proposition states the converse implication under suitable conditions.

Proposition 3.3. *Let $F : K \multimap X$ be a midpoint convex multimap with closed values. If F is upper semicontinuous from X to X_w, then F is convex.*

Proof. Let $t_x \in F(x)$, $t_y \in F(y)$ and $\lambda \in]0,1[$; we have to prove that $\lambda t_x + (1-\lambda)t_y \in F(x_\lambda)$, where $x_\lambda = \lambda x + (1-\lambda)y$.
We denote with $\mathbb{Q}_2$ the set of rational dyadic numbers. It is well known that if $\lambda \in]0,1[\cap \mathbb{Q}_2$ the midpoint convexity of F implies

$$\lambda F(x) + (1-\lambda)F(y) \subseteq F(x_\lambda). \tag{3.2}$$

Let $\{\lambda_k\} \subseteq]0,1[\cap \mathbb{Q}_2$ be a sequence converging to λ and set

$$\begin{aligned} \xi_k &= \lambda_k x + (1-\lambda_k)y, \\ t_k &= \lambda_k t_x + (1-\lambda_k)t_y, \\ t_\lambda &= \lambda t_x + (1-\lambda)t_y. \end{aligned}$$

Trivially $\xi_k \to x_\lambda$ and $t_k \to t_\lambda$.
By the upper semicontinuity of F, for all $V \in \mathcal{B}$ there exists $\delta_V > 0$ such that for all $x \in x_\lambda + \delta_V X_1$, it follows $F(x) \subset F(x_\lambda) + V$.
Since $\lambda_k \in \mathbb{Q}_2$ for any $k > 0$, by (3.2), we have that

$$t_k \in \lambda_k F(x) + (1-\lambda_k)F(y) \subseteq F(\xi_k).$$

Moreover there exists $k_0 > 0$ such that $\xi_k \in x_\lambda + \delta_V X_1$ for all $k \geq k_0$; hence

$$t_k \in F(\xi_k) \subset F(x_\lambda) + V \;\text{ for any } k \geq k_0.$$

Then by the convergence of $\{t_k\}$ to t_λ, we have that $t_\lambda \in \overline{F(x_\lambda) + V}$.
Now a standard Hahn-Banach argument proves that $t_\lambda \in F(x_\lambda)$. □

Corollary 3.1. *Let $F : K \multimap X$ be a midpoint convex multimap with closed values. If F is upper semicontinuous from X to X_w, then it is both strongly and weakly midpoint linear.*

The converse implication may fail to hold as the following example shows.

Example 3.1. Let $\varphi : [0, \infty) \to [0, 1]$ be any continuous non increasing map. Then, as shown in ([8]), the map $T : X \to X$ defined by $T(x) = \varphi(\|x\|)x$ is strongly midpoint linear.
Observe that any continuous map $f : K \to X$ is midpoint convex if and only if it is affine.
Then the map T is not midpoint convex, since it is not affine.

4. Fixed point theorems

In this section we shall obtain fixed point theorems for multivalued maps defined on a weakly compact domain. More precisely we shall prove two theorems: the first one (Theorem 4.2) gives the existence of fixed points, for upper semicontinuous α-nonexpansive maps, in the framework of Darbo's approach, while the second one (Theorem 4.3) follows the line of the classical Ky-Fan's Fixed Point Theorem, and applies when the domain is equipped with the weak topology and the target with the norm topology. In both theorems the suitable midpoint linearity holds in a weaker sense, stated in the following definition.

Definition 4.1. A multimap $F : K \multimap X$ is said to be *piecewise strongly midpoint linear* if for any $r > 0$ there exists $s(r) \in]0, r]$ and a finite decomposition of K, say $\mathcal{D}_r = \{D_1, \dots, D_n\}$, such that the sets $F_{s(r)} \cap D_i$ are convex for any $i = 1, \dots, n$.
Analogously we shall say that F is *piecewise weakly midpoint linear* provided for any $U \in \mathcal{B}$ there exists $V \in \mathcal{B}, V \subseteq U$ and a finite decomposition of K, say $\mathcal{D}_V = \{D_1, \dots, D_n\}$, such that the sets $F_{V(U)} \cap D_i$ are convex for any $i = 1, \dots, n$.

Remark 4.1. Note that the piecewise assumption may significantly increase the class of functions satisfying the previous definition (w.r.t. the mere strong or weak midpoint linearity).
For instance, whenever $X = \mathbb{R}$, every single valued continuous map on a convex set $I \subset \mathbb{R}$ admitting finitely many oscillations is piecewise midpoint linear. In fact, the map $g(x) = |x - f(x)|$ will also admit finitely many oscillations and hence we can decompose the domain into finitely many subintervals, say $D_1, \dots, D_n$ so that g is monotone in each D_j. Then one can take $\mathcal{D} = \{D_1, \dots, D_n\}$ for every $\varepsilon > 0$, (i.e. the choice of $\mathcal{D}$ does not depend upon ε) as in each D_i the restriction $f\lfloor_{D_i}$ is midpoint linear, because g is quasi convex in D_i (see [8] Proposition 3.7).

Remark 4.2. Observe that we only require the existence of a suitable decomposition of K, depending upon a subfamily of weak neighborhoods of zero. For instance, when X is a Banach lattice, one can check only those weak neighborhoods determined by positive functionals.

Throughout the remaining of this section, K will be a convex weakly compact subset of X and $F : K \multimap K$ will be a proper multimap with closed and convex values.
We begin with a necessary and sufficient condition.

Lemma 4.1. *Let F be piecewise strongly midpoint linear and upper semicontinuous from X to X_w. Then the following conditions are equivalent*

(a) *for any $r > 0$*

$$F_r = \{x \in K \ : \ x \in F(x) + rX_1\} \neq \emptyset; \tag{4.1}$$

(b) *F admits a fixed point.*

Proof. The converse being trivial, we only need to prove that (a) implies (b).
Since $r_1 < r_2$ implies that $F_{r_1} \subseteq F_{r_2}$, the family $\{\overline{F}_{s(r)},\ r > 0\}$ has the finite intersection property.
Let $r > 0$, by the piecewise strong midpoint linearity condition, there exists a finite decomposition, $\mathcal{D}_r = \{D_1, \ldots, D_n\}$, such that the sets $F_{s(r)} \cap D_i$ are convex for any $i = 1, \ldots, n$.
As

$$\overline{F}_{s(r)} = \overline{\bigcup_{i=1}^{n} [F_{s(r)} \cap D_i]} \subset \bigcup_{i=1}^{n} \overline{[F_{s(r)} \cap D_i]} \subset \overline{F}_{s(r)},$$

we have

$$\overline{F}_{s(r)} = \bigcup_{i=1}^{n} \overline{[F_{s(r)} \cap D_i]}.$$

Now each $\overline{F_{s(r)} \cap D_i}$ is closed and convex, and therefore weakly closed; this implies that $\overline{F}_{s(r)}$ is weakly closed too.
Therefore, K being weakly compact, $\bigcap_{r>0} \overline{F}_{s(r)} \neq \emptyset$. Obviously, as $s(r) \leq r$,

$$\emptyset \neq \bigcap_{r>0} \overline{F}_{s(r)} \subset \bigcap_{r>0} \overline{F}_r$$

and so we conclude that $\bigcap_{r>0} \overline{F}_r \neq \emptyset$.
Let $x_0 \in \bigcap_{r>0} \overline{F}_r$. We claim that $x_0 \in F(x_0)$.
Fix $V \in \mathcal{B}$, $V = V(x_1^*, \ldots, x_n^*, \varepsilon)$, and let $W = V\left(x_1^*, \ldots, x_n^*, \frac{\varepsilon}{4}\right)$.
There exists $r > 0$ such that $rX_1 \subset W$; since $x_0 \in \overline{F_r}$ and $F_r \subset F_W$, it follows $x_0 \in \overline{F_W}$.
Hence, there should exist a sequence $\{x_n\} \subset F_W$ converging to x_0.
By the upper semicontinuity of F, $F(x_n) \subset F(x_0) + W$, for $n \in \mathbb{N}$ suitably large and hence, as $x_n \in F_W$, that is $x_n \in F(x_n) + \overline{W}$, we have that $x_n \in F(x_0) + 2\overline{W}$ for $n \in \mathbb{N}$ suitably large.
Thus $x_0 \in \overline{F(x_0) + \overline{W}}$, and, by the weak compactness of $F(x_0)$, there follows that $x_0 \in F(x_0) + 2\overline{W}$.
In conclusion, $x_0 \in F(x_0) + \overline{V}$ for any $V \in \mathcal{B}$, which in turns implies that $x_0 \in F(x_0)$. □

Let α denote the usual Kuratovski measure of non compactness of X; namely for a bounded set $E \subset X$, $\alpha(E)$ is the infimum of all $\varepsilon > 0$ such that E admits a finite covering consisting of subsets with diameter less than ε. We refer to [3] for the properties of α.

Definition 4.2. A multimap $F : K \multimap X$ is said to be *α-nonexpansive* provided, for every bounded $E \subset K$,$\alpha(F(E)) \leq \alpha(E)$.
If $\alpha(F(E)) \leq \lambda\, \alpha(E)$ with $\lambda \in [0, 1[$, F is said to be *α-contractive.*

Theorem 4.1 ([12] Corollary 3.3.1.)**.** *Let $K \subset X$ be a convex, closed and bounded set, let $F : K \multimap K$ be an upper semicontinuous, α-contractive multimap, with closed and convex values. Then F admits a fixed point.*

Theorem 4.2. *Let F be piecewise strongly midpoint linear, upper semicontinuous and α-nonexpansive; then F admits a fixed point.*

Proof. We shall show that $F_r \neq \emptyset$ for any $r > 0$.
Note first that K is bounded : hence $K \subset RX_1$ for some $R > 0$. Let $x_0 \in K$ be fixed, $\lambda \in [0, 1[$ and define the multimap

$$F_\lambda(x) = (1 - \lambda)x_0 + \lambda F(x) = \{(1 - \lambda)x_0 + \lambda y\, ,\, y \in F(x)\}.$$

Let $E \subset K$ and consider $F_\lambda(E) := \bigcup_{x \in E} F_\lambda(x)$.
Clearly $F_\lambda(E) = (1 - \lambda)x_0 + \lambda F(E)$.
Thus

$$\alpha(F_\lambda(E)) = \alpha(\lambda F(E)) = \lambda\alpha(F(E)) \leq \lambda\alpha(E)$$

and since $\lambda < 1$ this means that F_λ is α-contractive. By Theorem 4.1, then, F_λ admits a fixed point $x_\lambda \in K$.
Let $x \in K$ and consider the usual Hausdorff distance

$$h(F_\lambda(x), F(x)) = \sup\{e\{F_\lambda(x), F(x)\}, e\{F(x), F_\lambda(x)\}\},$$

where

$$\begin{aligned} e\{F(x), F_\lambda(x)\} &= \sup_{y \in F(x)} d(y, F_\lambda(x)), \\ e\{F_\lambda(x), F(x)\} &= \sup_{y \in F_\lambda(x)} d(y, F(x)). \end{aligned}$$

Let us observe that for $y \in F(x)$,

$$\begin{aligned} d(y, F_\lambda(x)) &= \inf_{z \in F_\lambda(x)} \|y - z\| = \inf_{w \in F(x)} \|y - (1 - \lambda)x_0 - \lambda w\| \\ &\leq \|y - (1 - \lambda)x_0 - \lambda y\| \\ &= (1 - \lambda)\|y - x_0\| \leq (1 - \lambda)(R + \|x_0\|) \end{aligned}$$

Then

$$e\{F(x), F_\lambda(x)\} \leq (1 - \lambda)(R + \|x_0\|).$$

Now let $y \in F_\lambda(x)$, say $y = (1 - \lambda)x_0 + \lambda z$, with $z \in F(x)$. Then

$$\begin{aligned} d(y, F(x)) &\leq \|y - z\| = \|(1 - \lambda)x_0 + (\lambda - 1)z\| \\ &= (1 - \lambda)\|x_0 - z\| \leq (1 - \lambda)(R + \|x_0\|) \end{aligned}$$

Then

$$e\{F_\lambda(x), F(x)\} \le (1-\lambda)(R + \|x_0\|).$$

Hence

$$\lim_{\lambda\to 1} h(F(x), F_\lambda(x)) \le \lim_{\lambda\to 1}(1-\lambda)(R + \|x_0\|) = 0.$$

Moreover, since $d(x_\lambda, F(x_\lambda)) \le h(F_\lambda(x_\lambda), F(x_\lambda))$, one finds

$$\lim_{\lambda\to 1} d(x_\lambda, F(x_\lambda)) = 0,$$

and so

$$\inf\{d(x, F(x)) \,:\, x \in K\} = 0.$$

Therefore $F_r \neq \emptyset$ for any $r > 0$.
Also it is clear that if F is upper semicontinuous, then it is upper semicontinuous also when the target is equipped with the weak topology. Therefore we can apply Lemma 4.1, and obtain the claimed result. □

The single valued version of Theorem 4.2 reads as follows.

Corollary 4.1. *A α-nonexpansive, continuous, piecewise strongly midpoint linear map $f : K \to K$ admits at least a fixed point.*

The previous corollary slightly generalizes Theorem 4.5 of [8].
In the next result we obtain the existence of fixed points when the target space is equipped with the weak topology.

Theorem 4.3. *Let F be piecewise weakly midpoint linear and upper semicontinuous from X to X_w; then F admits a fixed point.*

Proof. We shall prove that

$$\bigcap_{U\in\mathcal{B}} F_U \neq \emptyset.$$

Clearly if $V \subset U$ then $F_V \subset F_U$. Therefore , letting $V = V(U) \in \mathcal{B}$ be determined by piecewise weakly midpoint linearity it is enough to prove that

$$\bigcap_{U\in\mathcal{B}} F_{V(U)} \neq \emptyset.$$

First of all we prove that for any $U \in \mathcal{B}$ the set $F_U \neq \emptyset$.
For the weak compactness of the set K there exist a finite numbers of points $\{z_1, \dots, z_n\} \in K$ such that $K \subset \bigcup_{i=1}^{n}(U + z_i)$. Denote with C the closed convex hull of $\{z_1, \dots, z_n\}$. Since K is convex it follows $C \subset K$. For any $x \in C$ define the map $\eta_U : C \multimap C$ as

$$\eta_U(x) = (F(x) + \overline{U}) \cap C.$$

The map η_U has nonempty convex and closed values.
We prove that η_U is upper semicontinuous from X to X_w.
Let $x_0 \in C$ and $\mathcal{U} \in \mathcal{B}$ be such that $\eta_U(x_0) \subset \mathcal{U}$.

Since $\eta_U(x_0)$ is compact it follows that there exists $U_1 \in \mathcal{B}$ such that $U_1 + \eta_U(x_0) \subset \mathcal{U}$. Moreover there exists $U_2 \in \mathcal{B}$ such that

$$(U_2 + F(x_0) + \overline{U}) \cap (U_2 + C) \subset U_1 + [(F(x_0) + \overline{U}) \cap C].$$

Then as $0 \in U_2$, one has

$$(U_2 + F(x_0) + \overline{U}) \cap C \subset U_1 + \eta_U(x_0) \subset \mathcal{U}.$$

Since F is upper semicontinuous from X to X_w, there exists a $r > 0$ such that

$$F(x) \subset U_2 + F(x_0) \quad \text{for any } x \in x_0 + rX_1 \cap K.$$

Then for $x \in x_0 + rX_1 \cap C$ we have

$$\eta_U(x) = (F(x) + \overline{U}) \cap C \subset (U_2 + F(x_0) + \overline{U}) \cap C \subset \mathcal{U}.$$

Then η_U is upper semicontinuous.
From Kakutani's fixed point Theorem (see [13]) there exists $x_0 \in C$ such that

$$x_0 \in \eta(x_0) \subset F(x_0) + \overline{U}.$$

Hence $F_U \neq \emptyset$, for any $U \in \mathcal{B}$.
Note that the family $\{F_{V(U)}, U \in \mathcal{B}\}$ has the finite intersection property. Indeed for any pair $U_1, U_2 \in \mathcal{B}$ one has

$$F_{V(U_1)} \cap F_{V(U_2)} \supseteq F_{V(U_1) \cap V(U_2)} \neq \emptyset,$$

where the last inequality follows from the previous step, since $V(U_1) \cap V(U_2) \in \mathcal{B}$.
Now we prove that for any $U \in \mathcal{B}$ the set $F_{V(U)} \neq \emptyset$ is closed with respect to the weak topology.
To this aim let $U \in \mathcal{B}$ be fixed; for the sake of simplicity we shall write F_V instead of $F_{V(U)}$.
We prove first that F_V is closed with respect to the strong topology.
Let $y \in K \setminus F_V$. Then $y \notin F(y) + \overline{V}$. Since $F(y) + \overline{V}$ is a weakly closed set there exists $V_3 \in \mathcal{B}$ such that

$$(y + V_3) \cap (F(y) + \overline{V} + V_3) = \emptyset. \tag{4.2}$$

Since F is upper semicontinuous from X to X_w there exists $\varepsilon > 0$ such that

$$F(z) \subset F(y) + V_3 \quad \text{for any } z \in (y + \varepsilon X_1) \cap K.$$

We can assume $\varepsilon X_1 \subset V_3$. Hence, from (4.2)

$$z \notin F(z) + \overline{V} \quad \text{for any } z \in (y + \varepsilon X_1) \cap K.$$

Otherwise, one would find $z \in F(z) + \overline{V} \subset F(y) + V_3 + \overline{V}$ and simultaneously $z \in (y + \varepsilon X_1) \subset y + V_3$, thus contradicting (4.2).
So $z \notin F_V$, for each $z \in y + \varepsilon X_1$. Hence F_V is closed with respect to the strong topology.
From the piecewise weak midpoint linearity, corresponding to $V \in \mathcal{B}$, we can find a finite decomposition of K, say $\mathcal{D}_V = \{D_1, \ldots, D_n\}$ such that $F_V \cap D_i$ is a convex set for any $i = 1, \ldots, n$. Clearly

$$F_V = \bigcup_{i=1}^{n} [F_V \cap D_i] \subseteq \bigcup_{i=1}^{n} \overline{[F_V \cap D_i]} \subseteq \overline{F_V} = F_V$$

and each $\overline{F_V \cap D_i}$ is closed with respect to the strong topology. On the other side, since $F_V \cap D_i$ is convex, $\overline{F_V \cap D_i}$ is also convex and therefore weakly closed.
Hence F_V is the union of finitely many weakly closed sets, and therefore itself weakly closed.
By the weak compactness of K it follows $\bigcap_{U \in \mathcal{B}} F_{V(U)} \neq \emptyset$.
Therefore there exists $q_0 \in \bigcap_{U \in \mathcal{B}} F_U$. Then, by a standard Hahn-Banach argument, $q_0 \in F(q_0)$. □

Theorem 4.3 applies in cases where Theorem 4.2 cannot be used as the following example shows.

Example 4.1. Let $X = c_o$ and let $\{t_p\}_p$ be a sequence in $\mathbb{R}$, with $t_p \downarrow t > 0$ for any $p \in \mathbb{N}$ and let

$$z_p = t_p \cdot \mathbf{e}_{p+1} \in c_o$$

where $\mathbf{e}_p$ is the generic element of the standard basis of c_o. Thus $\{z_p\}$ converges to 0 only with respect to the weak topology of c_o.
Let $\tau_p = t_p - t$ and $K_2 = \overline{\text{co}}\{0, z_p, p \in \mathbb{N}\} \subset c_o$.
By the Eberlein-Smulyan Theorem and Krein-Smulyan Theorem (see e.g. [9] Theorem 1 p. 430 and Theorem 4 p. 434), the set K_2 is weakly compact, so $K = [0, \tau_0] \times K_2 \subset c_o$ is convex and weakly compact in c_o.
We define the multimap $F : K \multimap K$ as follows:

$$F(x) = \begin{cases} \dfrac{x}{2} & \text{if} \quad \begin{array}{l} x_1 \neq \tau_p,\ p \in \mathbb{N}, \\ \text{or } x_1 = \tau_p \text{ and } x_j > 0 \text{ for some } j > 1; \end{array} \\ \left\{ \left(\dfrac{x_1}{2}, t \right),\ t \in \text{co}\{0, z_p\} \right\} & \text{if} \quad x_1 = \tau_p \text{ and } x_j = 0, \text{ for each } j > 1. \end{cases}$$

Claim 1. The multimap F is upper semicontinuous from X to X_w.
To prove it fix $\widetilde{x} \in K$ and $V = V(\varphi_1, \ldots, \varphi_n, \varepsilon) \in \mathcal{B}$, with $\varphi_i \in \ell^1$, $i = 1, \ldots, n$.

(i) Assume $\widetilde{x}_1 \neq \tau_p$, $p \in \mathbb{N}$. Then $F(\widetilde{x}) = \dfrac{\widetilde{x}}{2}$.
Observe that there exists a unique p_0 such that $\tau_{p_0+1} < \widetilde{x}_1 < \tau_{p_0}$.
Let $\delta_1 < \min\{\tau_{p_0} - \widetilde{x}_1; \widetilde{x}_1 - \tau_{p_0+1}\}$ and $x = \{x_n\} \in \widetilde{x} + \delta_1 X_1$. Then $x_1 \neq \tau_p$ for any $p \in \mathbb{N}$ and, therefore, $F(x) = \dfrac{x}{2}$. Thus in $\widetilde{x} + \delta_1 X_1$ the map F is strongly continuous and so it is continuous from X to X_w.

(ii) Assume now $\widetilde{x}_1 = \tau_p$, $p \in \mathbb{N}$ and $\widetilde{x}_j \neq 0$, for some $j > 1$; then again $F(\widetilde{x}) = \dfrac{\widetilde{x}}{2}$.
Let $\delta_2 < \max_{j>1} \widetilde{x}_j$ and $x = \{x_n\} \in \widetilde{x} + \delta_2 X_1$, then $x_j \neq 0$ for the same j and, therefore, $F(x) = \dfrac{x}{2}$. Thus, also in $\widetilde{x} + \delta_2 X_1$ the map F is strongly continuous and hence continuous from X to X_w.

(iii) Assume finally $\widetilde{x}_1 = \tau_p$, for some $p \in \mathbb{N}$ and $\widetilde{x}_j = 0$ for all $j > 1$; then

$$F(\widetilde{x}) = \left\{ \left(\frac{\widetilde{x}_1}{2}, t_0 \right),\ t_0 \in \text{co}\{0, z_p\} \right\}.$$

Let

$$\delta_3 < \frac{\varepsilon}{\max\limits_{1\le i\le n} \|\varphi_i\| \vee 1} \tag{4.3}$$

and let $x = \{x_n\} \in \widetilde{x} + \delta_3 X_1$.
If $x_1 \neq \tau_p$, or $x_1 = \tau_p$ and there exists $j > 1$ such that $x_j \neq 0$, then $F(x) = \dfrac{x}{2}$.
Clearly, as $\dfrac{\widetilde{x}}{2} \in F(\widetilde{x})$, we have

$$\left|\varphi_i\left(\frac{x}{2} - \frac{\widetilde{x}}{2}\right)\right| \le \frac{1}{2}\|\varphi_i\|\delta_3 < \varepsilon$$

and so $F(x) \in F(\widetilde{x}) + V$.
If $x_1 = \tau_p$ and $x_j = 0$ for all $j > 1$, then

$$F(x) = \left\{\left(\frac{x_1}{2}, t\right),\ t \in \mathrm{co}\{0, z_p\}\right\}.$$

Consider then $u = \left(\dfrac{x_1}{2}, \lambda z_p\right) \in F(x)$ with $\lambda \in (0, 1]$, and correspondingly

$$v = \left(\frac{\widetilde{x}_1}{2}, \lambda z_p\right) \in F(\widetilde{x}).$$

Again by (4.3)

$$|\varphi_i(u - v)| = \left|\varphi_i^1 \cdot \frac{x_1 - \widetilde{x}_1}{2}\right| \le \frac{|\varphi_i^1|\delta_3}{2} < \varepsilon,$$

i.e. $F(x) \subset F(\widetilde{x}) + V$.

Claim 2. The map F is not upper semicontinuous with respect to the strong topology in 0.
Indeed assume F upper semicontinuous at 0. Then for any $r > 0$ there should exist $\delta > 0$ such that for any $x \in \delta X_1$ it follows $F(x) \subset rX_1$. Consider the sequence in K defined as $x_p = (\tau_p, 0, 0, \dots)$. Since $\|x_p\|_{c_o} = \tau_p \to 0$, $\{x_p\}$ strongly converges to 0. Moreover $\left(\dfrac{\tau_p}{2}, z_p\right) \in F(x_p)$. But, as we have pointed out above, $\{z_p\}$ does not converge to 0 in the norm topology, obtaining a contradiction.

Claim 3. The map F is piecewise weakly midpoint linear.
Again, fix $V = V(\varphi_1, \dots, \varphi_n, \varepsilon)$ with $\varphi_i \in \ell^1$. As observed in Remark 4.2 w.l.o.g. we can suppose $\varphi_i \in \ell^1_+$, $i = 1, \dots, n$.
Since z_p weakly converges to 0 and $\tau_p \to 0$, $\bar{p} \in \mathbb{N}$ exists so that $\varphi_j(z_p) < \varepsilon$, $j = 1, \dots, n$ and $\tau_p < \dfrac{\varepsilon}{\max\limits_{1\le i\le n} \|\varphi_i\| \vee 1}$ for $p \ge \bar{p}$.
We shall define the decomposition $\mathcal{D}_V$.
Let first $D_1 = [0, \tau_{\bar{p}}] \times \overline{V}$. Observe that D_1 is a convex set. We shall prove that $D_1 \cap F_V$ is convex.
Consider any $x \in D_1 \cap K$, say $x = \{x_n\}$.

If $x_1 \neq \tau_p$, for any $p \in \mathbb{N}$, or if $x_1 = \tau_p$ and there exists $j > 1$ such that $x_j \neq 0$, as $x - F(x) = \dfrac{x}{2}$, we have

$$\varphi_i[x - F(x)] = \varphi_i\left(\frac{x}{2}\right) = \varphi_i^1 \cdot \frac{x_1}{2} + \sum_{k=2}^{\infty} \varphi_i^k \cdot \frac{x_k}{2} \leq \frac{1}{2}\left(\varphi_i^1 \tau_{\bar{p}} + \sum_{k=2}^{\infty} \varphi_i^k \cdot x_k\right) \leq \varepsilon,$$

(remember that if $x = (x_1, x_2, \dots) \in D_1$, by definition $(x_2, \dots) \in \overline{V}$). Thus $x \in F_V$.
In the case $x = (\tau_p, 0, 0, \dots) \in D_1 \cap K$, clearly $p \geq \bar{p}$ (for $\tau_p \in [0, \tau_{\bar{p}}]$). Since $F(x) = \left\{\left(\dfrac{\tau_p}{2}, t\right), \, t \in \mathrm{co}\{0, z_p\}\right\}$ we can choose $\left(\dfrac{\tau_p}{2}, 0, 0, \dots\right) \in F(x)$ obtaining

$$\varphi_i\left(x - \frac{x}{2}\right) \leq \frac{1}{2}\varphi_i^1 \tau_p < \varepsilon$$

and in conclusion $x \in F(x) + V$.
Hence we have that $D_1 \cap K \subset F_V \subset K$ and so $F_V \cap D_1 = D_1 \cap K$, therefore it is convex.

For $j = 1, \dots, n$, let now $V_j = V(\varphi_j, \varepsilon)$ and $D_{j+1} = [0, \tau_{\bar{p}}] \times (K_2 \setminus V_j)$.
Then each D_{j+1} is a convex set.
Note also that if $x = \{x_n\} \in D_{j+1}$, for some j, then $x_k \neq 0$ for some $k > 1$. Therefore $F(x) = \dfrac{x}{2}$ for every $x \in D_{j+1}$, $j = 1, \dots, n$.
Fix i and consider $x', x'' \in D_{i+1} \cap F_V$; we have then

$$F\left(\frac{x' + x''}{2}\right) = \frac{x' + x''}{4} = \frac{F(x') + F(x'')}{2}; \tag{4.4}$$

hence

$$\frac{x' + x''}{2} - F\left(\frac{x' + x''}{2}\right) = \frac{1}{2}\left\{[x' - F(x')] + [x'' - F(x'')]\right\} \in V. \tag{4.5}$$

Therefore $D_{j+1} \cap F_V$ is a convex set, for each $j = 1, \dots, n$.

Now for $i = 1, \dots \bar{p} - 1$ consider the three convex sets

$$D_i' =]\tau_{i+1}, \tau_i[\times K_2,$$

$$D_i'' = \{\tau_i\} \times (K_2 \setminus \{0\}),$$

and

$$D_i''' = \{(\tau_i, 0, \dots, 0)\},$$

(note that $K_2 \setminus \{0\}$ remains a convex set).
If $x \in D_i'$, $F(x) = \dfrac{x}{2}$, then one obtains (4.4) and therefore (4.5) also in this case, thus proving that the set $F_V \cap D_i'$ is a convex set for any $i, \dots, \bar{p} - 1$.
Let now x', $x'' \in F_V \cap D_i''$, then $F(x') = \dfrac{x'}{2}$ and $F(x'') = \dfrac{x''}{2}$. Moreover, since 0 is an extremal point for K, $\dfrac{x' + x''}{2} \neq 0$, namely, there exists $j > 1$ such that $\dfrac{x_j' + x_j''}{2} \neq 0$.
Then $F\left(\dfrac{x' + x''}{2}\right) = \dfrac{x' + x''}{4}$ and again by (4.4) and (4.5), we have that the set $F_V \cap D_i''$ is a convex set for any $i, \dots, \bar{p} - 1$.

Finally $D_i''' \cap F_V$ is either empty or it reduces to the singleton $\{(\tau_i, 0, \dots, 0)\}$, in both cases it is a convex set.
In conclusion $\mathcal{D}_V = \{D_1, D_2, \dots, D_n, D_i', D_i'',, D_i''' \; i = 1, \dots, \bar{p} - 1\}$ is the required decomposition of K.

5. Applications

In the whole section we denote with $X^n = X \times \cdots \times X$ n-times and with X_1^j the unit ball of the space X^j, $j = 1, \dots, n$.
We will apply the fixed point Theorem 4.3 to obtain the existence of a social equilibrium in a game with n players.
We briefly remind the usual Debreu model for the game.
Assume that there are *n players* (or *agents*) and that each player i chooses *strategies* from a set $\mathcal{A}_i$. Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ be the product of all the possible strategies. Let $\overline{\mathcal{A}}_i$ denote the residual product $\overline{\mathcal{A}}_i = \mathcal{A}_1 \times \cdots \times \mathcal{A}_{i-1} \times \mathcal{A}_{i+1} \times \cdots \mathcal{A}_n$. For any $a \in \mathcal{A}$, $a = (a_1, \dots, a_n)$, let $\bar{a}_i$ denote the corresponding element in $\overline{\mathcal{A}}_i$, namely $\bar{a}_i = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$. To each player i, a payoff function $f_i : \overline{\mathcal{A}}_i \times \mathcal{A}_i \to \mathbb{R}$ is associated.
Given $\bar{a}_i$, the response of the i-th player to $\bar{a}_i$ is bounded to lie in a subset $C_i(\bar{a}_i) \subset \mathcal{A}_i$. Therefore players want to maximize their payoff functions $f_i(\bar{a}_i, x)$ on the set $C_i(\bar{a}_i)$, that is player i wants to choose among the possible strategies $C_i(\bar{a}_i)$ those x's that maximize the payoff f_i.
This leads immediately to the following classical definition of social equilibrium.

Definition 5.1. A vector $a^* \in \mathcal{A}$ is a *social equilibrium point* if for any $i = 1, \dots, n$

$$a_i^* \in C_i(\bar{a}_i^*) \text{ and } f_i(a^*) = \max_{x \in C_i(\bar{a}_i^*)} f_i(\bar{a}_i^*, x).$$

To obtain a social equilibrium point we need the following result.

Theorem 5.1 (Corollary 9.2.6 [14])**.** *Let X and Y be topological spaces, $F : X \multimap Y$ be a lower semicontinuous multimap with compact values, and $\psi : X \times Y \to \mathbb{R}$ be a continuous map. Then the multimap $m' : X \multimap Y$ defined as*

$$m'(x) = \{\bar{y} \in F(x) \; : \; \psi(x, \bar{y}) = \max_{y \in F(x)} \psi(x, y)\}$$

is an upper semicontinuous compact valued multimap.

We state now the main result in this section.

Theorem 5.2. *Let $\mathcal{A}$ be a convex and $(X_w)^n$-compact set. Assume that*

- (A1) *$C_i : \overline{\mathcal{A}}_i \multimap \mathcal{A}_i$ is a midpoint convex lower semicontinuous from $(X)^{n-1}$ to X_w multimap with closed values;*
- (A2) *$f_i : \overline{\mathcal{A}}_i \times \mathcal{A}_i \to \mathbb{R}$ is midpoint concave and $(X^{n-1}, X_w)-$continuous;*
- (A3) *for each $a \in \mathcal{A}_i$, $a = (a_1, \dots, a_n)$ the map $\hat{f}_i(\bar{a}_i) = \sup_{\eta \in C_i(\bar{a}_i)} f_i(\bar{a}_i, \eta)$ is midpoint convex, namely for every $a, b \in \mathcal{A}$ the following relationship holds:*

$$\hat{f}_i\left(\frac{\overline{a}_i + \overline{b}_i}{2}\right) = \frac{1}{2}\left(\hat{f}_i(\overline{a}_i) + \hat{f}_i(\overline{b}_i)\right).$$

Then there exists a social equilibrium point.

Proof. We shall divide the proof in several steps.
Step 1. The multimap $M_i : \overline{\mathcal{A}}_i \multimap \mathcal{A}_i$ *defined as*

$$M_i(\bar{a}_i) = \left\{ x \in C_i(\bar{a}_i) \ : \ f_i(\bar{a}_i, x) = \hat{f}_i(\bar{a}_i) \right\}$$

has non empty values.
Indeed, by Proposition 3.2, $C_i(x)$ is convex for any $x \in \overline{\mathcal{A}}_i$, and hence weakly closed in $\mathcal{A}_i$; thus $C_i(x)$ is weakly compact.
From the weak compactness of $\mathcal{A}$ in X^n, by the Eberlain-Smulyan Theorem and by the Mazur convexity Theorem it is easy to show that f_i is a bounded map and that the Weierstrass Theorem holds.

Step 2. M_i *is a midpoint convex multimap.*
We have to prove that for any $i = 1, \dots, n$, $\overline{a}_i \in \overline{\mathcal{A}}_i$, $\overline{b}_i \in \overline{\mathcal{A}}_i$ M_i satisfies the following inclusion:

$$\frac{1}{2}[M_i(\overline{a}_i) + M_i(\overline{b}_i)] \subseteq M_i \left(\frac{\overline{a}_i + \overline{b}_i}{2} \right).$$

Let $c_i \in M_i(\overline{a}_i)$, $d_i \in M_i(\overline{b}_i)$ and $x_i = \dfrac{c_i + d_i}{2}$.
We have that $c_i \in C_i(\overline{a}_i)$ $d_i \in C_i(\overline{b}_i)$; then by (A1)

$$x_i \in \frac{1}{2}[C_i(\overline{a}_i) + C_i(\overline{b}_i)] \subseteq C_i \left(\frac{\overline{a}_i + \overline{b}_i}{2} \right).$$

Moreover

$$f_i(\overline{a}_i, c_i) = \max_{t \in C_i(\overline{a}_i)} f_i(\overline{a}_i, t)$$
$$f_i(\overline{b}_i, d_i) = \max_{t \in C_i(\overline{b}_i)} f_i(\overline{b}_i, z)$$

By (A2) and (A3) we obtain

$$f_i \left(\frac{\overline{a}_i + \overline{b}_i}{2}, x_i \right) \geq \frac{f_i(\overline{a}_i, c_i) + f_i(\overline{b}_i, d_i)}{2} = \max_{\eta \in C_i \left(\frac{\overline{a}_i + \overline{b}_i}{2} \right)} f_i \left(\frac{\overline{a}_i + \overline{b}_i}{2}, \eta \right).$$

Step 3. M_i *is upper semicontinuous from* X^{n-1} *to* X_w *with convex and closed values.*
In fact by Step 1 $C_i(x)$ is weakly compact for any $x \in \overline{\mathcal{A}}_i$; hence we can apply Theorem 5.1 with $E = X^{n-1}$ and $Y = X_w$ obtaining that M_i is upper semicontinuous from X^{n-1} to X_w with weakly compact, hence closed, values. Moreover by Step 2 and Proposition 3.2 $M_i(x)$ is convex for any $x \in \overline{\mathcal{A}}_i$.

Step 4. The multimap $\Phi : \mathcal{A} \multimap \mathcal{A}$ *defined as*

$$\Phi(a) = M_1(\bar{a}_1) \times M_2(\bar{a}_2) \times \cdots \times M_n(\bar{a}_n).$$

is $(X^n, (X_w)^n)$*-upper semicontinuous.*
Fix $V = V_1 \times \cdots \times V_n$ a basic neighbourhood of 0 in $(X_w)^n$. From Step 3 the multimaps M_i are (X^{n-1}, X_w)-upper semicontinuous for any $i = 1, \dots, n$. Hence

for any $V_i \in \mathcal{B}$ there exists $r > 0$ such that for any $\bar{a}_i \in \bar{\mathcal{A}}_i \cap (\bar{a}_0^i + rX_1^{n-1})$ it holds $M_i(\bar{a}_i) \subseteq M_i(\bar{a}_0^i) + V_i$. If $a \in \mathcal{A} \cap (a_0 + rX_1^n)$ then $\bar{a}_i \in \bar{\mathcal{A}} \cap (\bar{a}_0^i + rX_1^{n-1})$. Hence

$$\Phi(a) = M_1(\bar{a}_1) \times M_2(\bar{a}_2) \times \cdots \times M_n(\bar{a}_n) \subseteq \prod_{i=1}^{n} (M_i(\bar{a}_0^i) + V_i) \subseteq \Phi(a_0) + V.$$

Step 5. The multimap Φ is midpoint convex.
Let $u \in \dfrac{\Phi(a) + \Phi(b)}{2}$. Hence $u = \dfrac{u_1 + u_2}{2}$ with $u_1 \in \Phi(a)$, $u_2 \in \Phi(b)$, that is $u_1 = (u_1^1, \ldots, u_n^1)$ and $u_2 = (u_1^2, \ldots, u_n^2)$ with $u_i^1 \in M_i(\bar{a}_i)$ and $u_i^2 \in M_i(\bar{b}_i)$. By the midpoint convexity of the multimaps M_i, $i = 1, \ldots, n$, proved in Step 2 we obtain

$$\begin{aligned} u &= \left(\frac{u_1^1 + u_1^2}{2}, \ldots, \frac{u_n^1 + u_n^2}{2} \right) \\ &\in \left[\frac{M_1(\bar{a}_1) + M_1(\bar{b}_1)}{2} \right] \times \cdots \times \left[\frac{M_n(\bar{a}_n) + M_n(\bar{b}_n)}{2} \right] \\ &\subseteq M_1 \left[\frac{\bar{a}_1 + \bar{b}_1}{2} \right] \times \cdots \times M_n \left[\frac{\bar{a}_n + \bar{b}_n}{2} \right] = \Phi \left(\frac{a + b}{2} \right). \end{aligned}$$

Step 6. The multimap Φ is midpoint linear.
Indeed by Step 3 $\Phi(a)$ is the product of a finite number of weakly compact sets; therefore $\Phi(a)$ is $(X_w)^n$-compact, hence $(X_w)^n$-closed, so X^n-closed. The result then follows by Steps 4, 5 and Corollary 3.1.

Hence we can apply Theorem 4.3 obtaining a fixed point $\bar{a} \in \Phi(\bar{a})$. Finally a standard argument shows that $\bar{a}$ is a social equilibrium point. □

Remark 5.1. We observe that the hypothesis of lower semicontinuity of the multimap C_i can be assumed only on the boundary of the sets $\bar{\mathcal{A}}_i$. Indeed the midpoint convexity implies the lower semicontinuity from X^{n-1} to X_w of the multimaps C_i in the interior of $\bar{\mathcal{A}}_i$ for any $i = 1, \ldots, n$ (see [5] (Theorem 3.2)).

Corollary 5.1. *Let $\mathcal{A}$ be a convex and $(X_w)^n$-compact set. Assume that $C_i : \overline{\mathcal{A}}_i \multimap \mathcal{A}_i$ is an additive lower semicontinuous from X^{n-1} to X_w multimap with closed values and $f_i : \overline{\mathcal{A}}_i \times \mathcal{A}_i \to \mathbb{R}$ is a linear on convex combinations and continuous function. Then there exists an equilibrium point.*

Proof. Observe that conditions (A1) and (A2) of the previous Theorem are trivially fulfilled.
Let $a, b \in \mathcal{A}$ and $c_i \in C_i(\bar{a}_i)$, $d_i \in C_i(\bar{b}_i)$ be such that

$$\begin{aligned} f_i(\bar{a}_i, c_i) &= \max_{t \in C_i(\bar{a}_i)} f_i(\bar{a}_i, t) \\ f_i(\bar{a}_i, d_i) &= \max_{z \in C_i(\bar{b}_i)} f_i(\bar{b}_i, z). \end{aligned}$$

Let $x_i = \dfrac{c_i + d_i}{2} \in \dfrac{C_i(\bar{a}_i) + C_i(\bar{b}_i)}{2} = C_i\left(\dfrac{\bar{a}_i + \bar{b}_i}{2}\right)$. We have

$$\begin{aligned} & f_i\left(\frac{\bar{a}_i + \bar{b}_i}{2}, x_i\right) = f_i\left(\frac{\bar{a}_i + \bar{b}_i}{2}, \frac{c_i + d_i}{2}\right) = f_i\left[\frac{(\bar{a}_i, c_i) + (\bar{b}_i, d_i)}{2}\right] \\ & = \frac{1}{2}\left[f_i((\bar{a}_i, c_i) + f_i(\bar{b}_i, d_i)\right] \geq \frac{1}{2}\left[f_i((\bar{a}_i, t) + f_i(\bar{b}_i, z)\right] \end{aligned} \tag{5.1}$$

for any $t \in C_i(\bar{a}_i)$ and $z \in C_i(\bar{b}_i)$. Moreover

$$\frac{t+z}{2} \in \frac{C_i(\bar{a}_i) + C_i(\bar{b}_i)}{2} = C_i\left[\frac{\bar{a}_i + \bar{b}_i}{2}\right],$$

and for any $h \in C_i\left[\dfrac{\bar{a}_i + \bar{b}_i}{2}\right]$ there exist $t \in C_i(\bar{a}_i), z \in C_i(\bar{b}_i)$ such that $h = \dfrac{t+z}{2}$. From (5.1) it follows then

$$f_i\left(\frac{\bar{a}_i + \bar{b}_i}{2}, x_i\right) \geq f_i\left(\frac{\bar{a}_i + \bar{b}_i}{2}, h\right),$$

for any $h \in C_i\left[\dfrac{\bar{a}_i + \bar{b}_i}{2}\right]$. Hence

$$\begin{aligned} & \max_{h \in C_i\left[\frac{\bar{a}_i + \bar{b}_i}{2}\right]} f_i\left(\frac{\bar{a}_i + \bar{b}_i}{2}, h\right) = f_i\left(\frac{\bar{a}_i + \bar{b}_i}{2}, x_i\right) \\ & = \frac{f_i(\bar{a}_i, c_i) + f_i(\bar{b}_i, d_i)}{2} = \frac{\max_{t \in C_i(\bar{a}_i)} f_i(\bar{a}_i, t) + \max_{z \in C_i(\bar{b}_i)} f_i(\bar{a}_i, z)}{2}, \end{aligned}$$

that is condition (A3) of the previous theorem. □

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