Fixed Point Theory, Volume 9, No. 1, 2008, 275-291
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

MINIMAX THEOREMS FOR FUNCTIONS INVOLVING A REAL VARIABLE AND APPLICATIONS

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Abstract. In this paper, we offer an overview of various applications of certain minimax theorems for functions of two variables one of which runs over a real interval.

Key Words and Phrases: Minimax theorems, connectedness, integral functionals on L^p and Sobolev spaces, local and global minima, potential operators, critical points, nonlinear equations, multiplicity, singular set, bifurcation, boundary value problems, well-posedness of constrained minimization problems, Chebyshev sets.
2000 Mathematics Subject Classification: 49K35, 49K40, 90C47, 47H10, 47J15, 47J30, 58E05, 58E07, 34B15, 35J65, 54D05.

Let X, Y be two non-empty sets and $f : X \times Y \to \mathbf{R}$ a given function.

The object of minimax theory, in its classical sense, is to find conditions on X, Y and f which are sufficient to guarantee the validity of the equality

$$\sup_{y\in Y}\inf_{x\in X} f(x,y) = \inf_{x\in X}\sup_{y\in Y} f(x,y) \ . \tag{1}$$

The aim of this paper is to review some minimax theorems which share the following basic assumption: Y is a real interval. More properly, our primary aim is to give an overview of the various applications of some of these theorems.

This paper was presented at the International Conference on Nonlinear Operators, Differential Equations and Applications held in Cluj-Napoca (Romania) from July 4 to July 8, 2007.

Of course, assuming that Y is a real interval is a severe restriction. But, once we pay such a price, we are then allowed to assume conditions on $f(\cdot, y)$ which are extremely more general than those appearing in the results where Y is, on the contrary, of general nature.

Let us start with our review.

Probably, the first minimax theorem belonging to the class we are dealing with is due to O. Yu. Borenshtein and V. S. Shul'man ([11]). The statement is as follows:

Theorem 1. *Let X be a compact metric space and let $Y \subseteq \mathbf{R}$ be an interval. Assume that f is continuous in $X \times Y$ and that*
(i) for each $y \in Y$, each local minimum of $f(\cdot, y)$ is a global minimum;
(ii) for each $x \in X$, $f(x, \cdot)$ is concave in Y.

Then, equality (1) *holds.*

Though Theorem 1 is affected by two rather heavy assumptions (that is to say, compactness of X and joint continuity of f in $X \times Y$), the authors were able to apply it to give a new proof of the following remarkable result by E. Asplund and V. Pták ([1]):

Theorem 2. *Let H be a real Hilbert space and $A, B : H \to H$ two continuous linear operators.*

Then, for every interval $I \subseteq \mathbf{R}$, one has

$$\inf_{\lambda \in I} \|A - \lambda B\|_{\mathcal{L}(H)} = \sup_{\|x\|=1} \inf_{\lambda \in I} \|A(x) - \lambda B(x)\| \ .$$

We will come back to Theorem 1 later.

Continuing in chronological order, we find the following result ([47]):

Theorem 3. *Let X be a topological space and Y a compact real interval. Assume that, for each $\rho \in \mathbf{R}$, $x_0 \in X$, $y_0 \in Y$, the sets*

$$\{x \in X : f(x, y_0) \leq \rho\}$$

and

$$\{y \in Y : f(x_0, y) > \rho\}$$

are connected. In addition, assume that at least one of the following three sets of conditions is satisfied:
(h_1) $f(x, \cdot)$ is upper semicontinuous in Y for each $x \in X$, and $f(\cdot, y)$ is lower semicontinuous in X for each $y \in Y$;

(h_2) *f is upper semicontinuous in $X \times Y$;*

(h_3) *X is compact, and f is lower semicontinuous in $X \times Y$.*

Then, equality (1) *holds.*

Theorem 3 has successfully been applied to integral functionals on L^p spaces.

Precisely, let $(T, \mathcal{F}, \mu)$ be a σ-finite non-atomic measure space, E a real Banach space ($E \neq \{0\}$), and p a real number greater than or equal to 1.

Let $L^p(T, E)$ denote the space of all (equivalence classes of) strongly μ-measurable functions $u : T \to E$ such that

$$\int_T \| u(t) \|^p \, d\mu < +\infty$$

equipped with the norm

$$\| u \|_{L^p(T,E)} = \left(\int_T \| u(t) \|^p \, d\mu \right)^{\frac{1}{p}} .$$

A set $D \subseteq L^p(T, E)$ is said to be decomposable if, for every $u, v \in D$ and every $S \in \mathcal{F}$, the function

$$t \to \chi_S(t)u(t) + (1 - \chi_S(t))v(t)$$

belongs to S, where χ_S denotes the characteristic function of S.

A function $\varphi : T \times E \to \mathbf{R}$ is said to be sup-measurable if for every strongly μ-measurable function $u : T \to E$, the function $t \to \varphi(t, u(t))$ is μ-measurable.

In [67], J. Saint Raymond established the following very interesting result:

Theorem 4. *Let $\varphi : T \times E \to \mathbf{R}$ be a sup-measurable function, and let $D \subseteq L^p(T, E)$ be a decomposable set.*

Then, if we put

$$S = \{u \in D : \varphi(\cdot, u(\cdot)) \in L^1(T)\},$$

for each $\rho \in \mathbf{R}$, the set

$$\left\{ u \in S : \int_T \varphi(t, u(t)) d\mu \leq \rho \right\}$$

is arcwise connected.

Then, applying Theorem 3 via Theorem 4, we get

Theorem 5. *Let $Y \subseteq \mathbf{R}$ be a compact interval, $X \subseteq L^p(T, E)$ a decomposable set, $\varphi : T \times E \times Y \to \mathbf{R}$ a function which is sup-measurable in $T \times E$, and concave in Y. Moreover, assume that $\varphi(\cdot, u(\cdot), y) \in L^1(T)$ for all $u \in X$, $y \in Y$.*

Finally, suppose that the functional $u \to \int_T \varphi(t, u(t), y)d\mu$ *is lower semicontinuous in* X *for each* $y \in Y$, *and that the function* $y \to \int_T \varphi(t, u(t), y)d\mu$ *is upper semicontinuous in* Y *for each* $u \in X$.

Then, one has

$$\sup_{y\in Y}\inf_{u\in X}\int_T \varphi(t, u(t), y)d\mu = \inf_{u\in X}\sup_{y\in Y}\int_T \varphi(t, u(t), y)d\mu.$$

From Theorem 5, in turn, many consequences follow. Let us here recall some of them ([48]-[51]).

Theorem 6. *Let* $\varphi : T \times E \to \mathbf{R}$ *be a sup-measurable function. Assume that there exist* $\alpha \in L^1(T)$, $\gamma_i \in]0, 1[$ *and* $\beta_i \in L^{\frac{p}{p-\gamma_i}}(T)$ $(i = 1, ..., k)$ *such that*

$$-\alpha(t) \le \varphi(t, x) \le \alpha(t) + \sum_{i=1}^{k} \beta_i(t) \parallel x \parallel^{\gamma_i}$$

for almost every $t \in T$ *and for every* $x \in E$.

Then, for every decomposable linear subspace X *of* $L^p(T, E)$ *and every closed hyperplane* V *of* X, *one has*

$$\inf_{u\in V}\int_T \varphi(t, u(t))d\mu = \inf_{u\in X}\int_T \varphi(t, u(t))d\mu\ .$$

Let us now observe a consequence of Theorem 6 which extends the classical fact that, for $\gamma \in]0, 1[$, the topological dual of $L^\gamma(T, E)$ reduces to zero. Precisely, we denote by $\mathcal{M}$ the set of all metrics d on $L^p(T, E)$ of the following type:

$$d(u, v) = \sum_{i=1}^{k}\int_T \beta_i(t) \parallel u(t) - v(t) \parallel^{\gamma_i} d\mu$$

where $u, v \in L^p(T, E)$, $\gamma_i \in]0, 1[$, $\beta_i \in L^{\frac{p}{p-\gamma_i}}(T)$, $\beta_i > 0$ in T $(i = 1, ..., k)$. Note that each $d \in \mathcal{M}$ is a metric inducing a vector topology which is weaker than the $\parallel \cdot \parallel_{L^p(T,E)}$-topology.

Theorem 7. *For every* $d \in \mathcal{M}$ *and every decomposable linear subspace* X *of* $L^p(T, E)$, *the topological dual of* (X, d) *reduces to zero.*

When we take $X = L^p(T, E)$, the conclusion of Theorem 6 can be extended to a class of functions φ with a more general growth.

Theorem 8. *Let* $\varphi : T \times E \to [0, +\infty[$ *be such that* $\varphi(\cdot, x)$ *is* μ*-measurable for each* $x \in E$ *and* $\varphi(t, \cdot)$ *is Lipschitzian with Lipschitz constant* $M(t)$ *for* *almost every $t \in T$, where $M \in L^{\frac{p}{p-1}}(T)$. Assume that $\varphi(\cdot, 0) \in L^1(T)$ and that there exists a sequence $\{\lambda_n\}$ in $]0, +\infty[$, with $\lim_{n\to+\infty} \lambda_n = +\infty$, such that, for almost every $t \in T$ and for every $x \in E$, one has*

$$\lim_{n\to+\infty} \frac{\varphi(t, \lambda_n x)}{\lambda_n} = 0.$$

Then, for every closed hyperplane V of $L^p(T, E)$, one has

$$\inf_{u\in V} \int_T \varphi(t, u(t))d\mu = \inf_{u\in L^p(T,E)} \int_T \varphi(t, u(t))d\mu.$$

Let us recall that a multifunction $F : T \to 2^E$ is said to be measurable if, for every open set $\Omega \subseteq E$, one has $\{t \in T : F(t) \cap \Omega \neq \emptyset\} \in \mathcal{F}$. A function $u : T \to E$ is a selection of the multifunction $F : T \to 2^E$ if $u(t) \in F(t)$ for all $t \in T$. We denote by $\mathcal{S}_F$ the set of all selections of F belonging to $L^1(T, E)$.

An application of Theorem 8 gives

Theorem 9. *Let E be separable, and let $F : T \to 2^E$ be a measurable multifunction, with non-empty closed values. Assume that* $\mathrm{dist}(0, F(\cdot)) \in L^1(T)$ *and that there exists a sequence $\{\lambda_n\}$ in $]0, +\infty[$, with $\lim_{n\to+\infty} \lambda_n = +\infty$, such that, for almost every $t \in T$ and for every $x \in E$, one has*

$$\lim_{n\to+\infty} \frac{\mathrm{dist}(\lambda_n x, F(t))}{\lambda_n} = 0.$$

Then, $\mathcal{S}_F$ intersects each closed hyperplane of $L^1(T, E)$.

Other related papers are [45], [46] and [27].

Now, come back to Theorem 1. In [68], J. Saint Raymond improved it weakening, in particular, the joint continuity assumption on f. Indeed, he obtained the following

Theorem 10. *Let X, Y be as in Theorem 1, and let f satisfy the following conditions:*

(b_1) for every $x \in X$, the function $f(x, \cdot)$ is concave and continuous;

(b_2) for every $y \in Y$, the function $f(\cdot, y)$ is lower semicontinuous;

(b_3) there exists a set $D \subseteq Y$, dense in Y, such that, for every $y \in D$, each local minimum of the $f(\cdot, y)$ is a global minimum.

Then, equality (1) *holds.*

In [52], we applied Theorem 10 to get the following result:

Theorem 11. *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbf{R}$ a sequentially weakly lower semicontinuous C^1 functional whose derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbf{R}$ a C^1 functional with compact derivative; $I \subseteq \mathbf{R}$ an interval. Assume that*

$$\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty$$

for all $\lambda \in I$, and that there exists a continuous concave function $h : I \to \mathbf{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) \ .$$

Then, there exist an open interval $J \subseteq I$ and a positive real number ρ such that, for each $\lambda \in J$, the equation

$$\Phi'(x) + \lambda\Psi'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

In turn, Theorem 11 (jointly with its variants obtained in [7] and in [43]) proved itself to be one of the most used results in the last years for the study of multiple solutions of nonlinear boundary value problems (see, for instance, [2]-[10], [18]-[20], [28]-[38], [40]-[44], [69],[70], [73]-[75]).

In [54], we improved Theorem 1 not only weakening the joint continuity assumption on f (as Saint Raymond did with Theorem 10), but also supposing that X simply is a topological space. Indeed, we obtained the following

Theorem 12. *Let X be a topological space, $Y \subseteq \mathbf{R}$ an interval, and let f satisfy the conditions:*

(i) for every $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous;

(ii) for every $y \in Y$, the function $f(\cdot, y)$ is lower semicontinuous and each of its local minima is a global minimum;

(iii) there exist $\rho > \sup_Y \inf_X f$ and $y_0 \in Y$ such that the set

$$\{x \in X : f(x, y_0) \le \rho\}$$

is compact.

Then, equality (1) *holds.*

Theorem 12 was then applied in [56], jointly with a recent result of I. G. Tsar'kov on Chebyshev sets ([72]), to get the following general multiplicity theorem:

Theorem 13. *Let X be a real Hilbert space and let $J : X \to \mathbf{R}$ be a C^1 nonconstant functional, with compact derivative, such that*

$$\liminf_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} \geq 0 \ .$$

Then, for each $r \in]\inf_X J, \sup_X J[$ for which the set $J^{-1}(]-\infty, r])$ is not convex and for each convex set $S \subseteq X$ dense in X, there exist $x_0 \in S \cap J^{-1}(]r, +\infty[)$ and $\lambda > 0$ such that the equation

$$x + \lambda J'(x) = x_0$$

has at least three solutions.

Theorem 13 has been extended by F. Faraci and A. Iannizzotto ([21]) to a more general class of Banach spaces. Some applications of the method introduced in [56] to differential equations, hemivariational inequalities and discrete boundary value problems can be found in [12], [13], [22], [24], [33].

Let H be a real Hilbert space.

As usual, for a generic operator $T : H \to H$, we say that T is a local homeomorphism at a point $x_0 \in H$ if there are a neighbourhood U of x_0 and a neighbourhood V of $T(x_0)$ such that the restriction of T to U is a homeomorphism between U and V. If T is not a local homeomorphism at x_0, we say that x_0 is a singular point of T.

The set of all singular points of T is called the singular set of T and we denote it by S_T. Clearly, the set S_T is closed.

In [64], combining the ideas of [56] (so, in particular, making an essential use of Theorem 12) with a remarkable result by R. S. Sadyrkhanov ([66]), we obtained the following

Theorem 14. *Let H be an infinite-dimensional real Hilbert space and let $J : H \to \mathbf{R}$ be a C^1 functional. Assume that J is sequentially weakly lower semicontinuous, not quasi-convex, and positively homogeneous of degree $\alpha \neq 2$. If $\alpha > 2$ assume also that J is non-negative. Denote by T the operator $x \to x + J'(x)$. Suppose that T is closed.*

Then, both the sets S_T and $T(S_T)$ are not σ-compact.

For instance, an application of Theorem 14 gives the following bifurcation result:

Theorem 15. *Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain, let $\beta \in L^\infty(\Omega)$, with* $\operatorname{ess\,sup}_\Omega \beta > 0$*, and let $q \in]0, 1[$.*

For each $\varphi \in H_0^1(\Omega)$, denote by Λ_φ the set of all weak solutions of the problem

$$\begin{cases} -\Delta u = \beta(x)|u+\varphi(x)|^{q-1}(u+\varphi(x)) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

Then, there exist two closed, not σ-compact sets $A, B \subset H_0^1(\Omega)$ with the following properties:
(i) for each $\varphi \in B$ there exist $w \in A$ and three sequences $\{u_k\}$, $\{v_k\}$ and $\{\varphi_k\}$ in $H_0^1(\Omega)$ such that

$$\lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k = w - \varphi, \quad \lim_{k\to\infty} \varphi_k = \varphi$$

and, for each $k \in \mathbf{N}$,

$$u_k \neq v_k \ \text{and} \ u_k, v_k \in \Lambda_{\varphi_k} \ ;$$

(ii) for each $\varphi \in H_0^1(\Omega) \setminus B$, the set Λ_φ is non-empty, finite and disjoint from $A - \varphi$.

Another application of Theorem 14 can be found in [26].

Recently, in [58], [63], we revisited Theorem 3, proving the following

Theorem 16. *Let X be a topological space, $Y \subseteq \mathbf{R}$ an interval and let $f(x,\cdot)$ be upper semicontinuous for each $x \in X$. Assume that there exist a number $\rho^* > \sup_Y \inf_X f$, a point $\hat{y} \in Y$ and a set $D \subseteq Y$, dense in Y, such that for each $\rho \in]-\infty, \rho^*[$, the following conditions hold:*
(i) the set $\{y \in Y : f(x,y) > \rho\}$ is an interval for all $x \in X$;
(ii) the set $\{x \in X : f(x,y) \leq \rho\}$ is closed for all $y \in Y$ and compact for $y = \hat{y}$, while the set $\{x \in X : f(x,y) < \rho\}$ is connected for all $y \in D$.

Then, equality (1) *holds.*

Here is a consequence of Theorem 16.

Theorem 17. *Let X be a reflexive real Banach space and Y a real interval. Assume that $f(x,\cdot)$ is concave in Y for all $x \in X$ and that $f(\cdot,y)$ is continuous, coercive and sequentially weakly lower semicontinuous in X for all $y \in Y$. Further, assume that*

$$\sup_Y \inf_X f < \inf_X \sup_Y f \ .$$

Then, for each $\rho > \sup_Y \inf_X f$, there exist a non-empty open set $A \subseteq Y$ with the following property: for every $\lambda \in A$ and every sequentially weakly lower semicontinuous functional $g : X \to \mathbf{R}$, there exists $\delta > 0$, such that, for each

$\mu \in [0, \delta]$, the functional $f(\cdot, \lambda) + \mu g(\cdot)$ has at least two local minima lying in the set $\{x \in X : f(x, \lambda) < \rho\}$.

There are already several papers where Theorem 17 has been applied to differential problems (see [14]-[17], [23], [39]).

For instance, one obtains results of this kind:

Theorem 18. *Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a continuous function and let $F(x) = \int_0^x \varphi(t)dt$. Assume that*

$$\limsup_{|x| \to +\infty} \frac{F(x)}{x^2} \leq 0$$

and that there is $r > 0$ such that

$$\sup_{|x| \leq 2\sqrt{r}} F(x) < 2r \sup_{|x| > \sqrt{2r}} \frac{F(x)}{x^2} \ .$$

Then, there exist $\rho > 0$ and a non-empty open set $A \subset]0, +\infty[$ with the following property: for each $\lambda \in A$ and for each continuous function $g : [0, 1] \times \mathbf{R} \to \mathbf{R}$, there exists $\delta > 0$ such that, for every $\mu \in [0, \delta]$, the problem

$$\begin{cases} -u'' + u = \lambda\varphi(u) + \mu g(t, u) & in \quad [0, 1] \\ u'(0) = u'(1) = 0 \end{cases}$$

has at least three classical solutions whose norms in $H^1(0, 1)$ are less than ρ.

Finally, let us arrive to the last minimax theorem of the present review. It reads as follows ([65]):

Theorem 19. *Let X be a Hausdorff topological space and $Y \subseteq \mathbf{R}$ be an interval. Assume that there exist a number $\rho^* > \sup_Y \inf_X f$ and a point $\hat{y} \in Y$ such that, for each $\rho \leq \rho^*$, the following conditions hold:*

(i) the set $\{y \in Y : f(x, y) > \rho\}$ is connected for all $x \in X$;

(ii) the set $\{x \in X : f(x, y) \leq \rho\}$ is sequentially closed for all $y \in Y$ and sequentially compact for $y = \hat{y}$;

(iii) for each compact interval $I \subseteq Y$ for which $\sup_I \inf_X f < \rho$, there exists a continuous function $\varphi : I \to X$ such that $f(\varphi(y), y) < \rho$ for all $y \in I$.

Then, equality (1) *holds.*

Theorem 19 has been the main tool that we have used to obtain a very general well-posedness result: Theorem 20 below.

Before stating it, let us introduce some notation.

Let X be a Hausdorff topological space, let a, b be two numbers in $[-\infty, +\infty]$, with $a < b$, and let $J, \Phi : X \to \mathbf{R}$ be two given functions.

If $a \in \mathbf{R}$ (resp. $b \in \mathbf{R}$), we denote by M_a (resp. M_b) the set of all global minima of the function $J + a\Phi$ (resp. $J + b\Phi$), while if $a = -\infty$ (resp. $b = +\infty$), M_a (resp. M_b) stands for the empty set. We adopt the conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$.

We set

$$\alpha := \max \left\{ \inf_X \Phi, \sup_{M_b} \Phi \right\} ,$$

$$\beta := \min \left\{ \sup_X \Phi, \inf_{M_a} \Phi \right\} .$$

A usual, given a function $\varphi : X \to \mathbf{R}$ and a set $C \subseteq X$, we say that the problem of minimizing φ over C is well-posed if the following two conditions hold:

- the restriction of φ to C has a unique global minimum, say $\hat{x}$;
- every sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \varphi(x_n) = \inf_C \varphi$ converges to $\hat{x}$.

We then have

Theorem 20. *Assume that $\alpha < \beta$ and that, for each $\lambda \in]a, b[$, the function $J + \lambda\Phi$ has sequentially compact sub-level sets and admits a unique global minimum in X.*

Then, for each $r \in]\alpha, \beta[$, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed.

Moreover, if we denote by $\hat{x}_r$ the unique global minimum of $J_{|\Phi^{-1}(r)}$ ($r \in]\alpha, \beta[$), the functions $r \to \hat{x}_r$ and $r \to J(\hat{x}_r)$ are continuous in $]\alpha, \beta[$.

When $a = 0$ and $b = +\infty$, it is also interesting to reformulate Theorem 20 in terms of an alternative, stressing, in this way, the variety of its possible uses.

Theorem 21. *Assume that, for each $\lambda > 0$, the function $J + \lambda\Phi$ has sequentially compact sub-level sets.*

Then, at least one of the following assertions holds:

(i) J has at least one global minimum .

(ii) There exists $\lambda^* > 0$ such that the function $J + \lambda^*\Phi$ has at least two global minima .

(iii) For each $r \in] \inf_X \Phi, \sup_X \Phi[$, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed and, if $\hat{x}_r$ denotes the unique global minimum of $J_{|\Phi^{-1}(r)}$, the functions $r \to \hat{x}_r$ and $r \to J(\hat{x}_r)$ are continuous in $] \inf_X \Phi, \sup_X \Phi[$.

Theorem 20 is the definitive abstract result coming out from the minimax method that we had previously developed in specific settings ([59]-[62]). In these latter papers, for instance, we obtained the following two results. The symbol $B(x,r)$ (resp. $S(x,r)$) stands for the closed ball (resp. the sphere) of radius r centered at x.

Theorem 22. *Let X be a real Hilbert space and let $J : X \to \mathbf{R}$ be a C^1 functional with locally Lipschitzian derivative.*

Then, for each $x_0 \in X$ with $J'(x_0) \neq 0$, there exists $\delta > 0$ such that, for every $r \in]0, \delta[$, one has

$$\inf_{B(x_0,r)} J = \inf_{S(x_0,r)} J$$

and the problems of minimizing J over $S(x_0, r)$ and over $B(x_0, r)$ are well-posed.

Moreover, if we denote by $\hat{x}_r$ the unique global minimum of $J_{|S(x_0,r)}$ ($r \in]0, \delta[$), the function $r \to \hat{x}_r$ is Lipschitzian in $]0, \delta[$.

Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain.

The space $H^1(\Omega)$ is endowed with the usual norm

$$\|u\| = \left(\int_\Omega (|\nabla u(x)|^2 + |u(x)|^2) dx \right)^{\frac{1}{2}} .$$

Theorem 23. *Let $\varphi : \mathbf{R}^{n+1} \to \mathbf{R}$ be a C^1 function whose gradient is non-constant and Lipschitzian (with respect to the Euclidean metric), and let V be a closed linear subspace of $H^1(\Omega)$ containing a v_0 such that*

$$\int_\Omega (\varphi_\xi(0) v_0(x) + \nabla_\eta \varphi(0) \nabla v_0(x)) dx \neq 0 .$$

Denote by M_L the set (possibly empty) of all global minima of the restriction to V of the functional

$$u \to \frac{L}{2} \|u\|^2 + \int_\Omega \varphi(u(x), \nabla u(x)) dx ,$$

where L is the Lipschitz constant of $\nabla \varphi$.

Then, $0 \notin M_L$ and for every $r \in]0, \operatorname{dist}(0, M_L)[$ the problem of minimizing the functional

$$u \to \int_\Omega \varphi(u(x), \nabla u(x)) dx$$

over the sphere $\{u \in V : \|u\| = r\}$ is well-posed.

F. Faraci and A. Iannizzotto in [25], using the ideas of [62], proved the following best approximation result:

Theorem 24. *Let X be a real Hilbert space, let C be a non-empty subset of X. For each $r > 0$, set*

$$A_r = \{x \in X : \mathrm{dist}(x, C) \geq r\} \ .$$

Then, for every $y \in X \setminus \overline{\mathrm{conv}}(C)$ and every $r > 0$, the problem of minimizing the function

$$x \to \|x - y\|$$

over A_r is well-posed.

In connection with Theorem 24, the following problem arises:

PROBLEM 1. Is there a subset C of X such that, for some $r > 0$, the set A_r is not convex and, for every $y \in \overline{\mathrm{conv}}(C)$, there is a unique $\hat{x} \in A_r$ for which

$$\|\hat{x} - y\| = \mathrm{dist}(y, A_r) \ ?$$

In view of Theorem 24, a positive answer to Problem 1 would provide an example of a not convex Chebyshev set in a Hilbert space, solving a classical problem still open.

Here are the two last results of this review.

Let us recall that a set in a topological space is said to be totally disconnected if each of its connected components is a singleton.

A joint application of Theorem 21 with Theorem 2 of [55] gives

Theorem 25. *Let X be a real Hilbert space and let $J : X \to \mathbf{R}$ be a C^1 functional, with compact derivative, such that*

$$\liminf_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} \geq 0 \ .$$

Then, at least one of the following assertions holds:

(a) J has at least one global minimum.

(b) There exists $\lambda^* > 0$ such that, for every C^1 functional $\Phi : X \to \mathbf{R}$, with compact derivative, satisfying

$$\liminf_{\|x\| \to +\infty} \frac{\Phi(x)}{\|x\|^2} > -\infty \ ,$$

there exists $\delta > 0$ *such that, for each* $\mu \in [0, \delta]$, *the equation*

$$x + \lambda^* J'(x) + \mu \Phi'(x) = 0$$

has at least three solutions.

(c) *There exists* $\hat{\lambda} > 0$ *such that the set of all global minima of the functional*

$$x \to \|x\|^2 + \hat{\lambda} J(x)$$

is not totally disconnected.

(d) *For each* $r > 0$, *the problem of minimizing* J *over* $S(0, r)$ *is well-posed.*

Finally, a joint application of Theorem 21 with Theorem 1 of [57] gives

Theorem 26. *Let* $\varphi : \mathbf{R} \to \mathbf{R}$ *be a continuous function such that, for each* $\lambda > 0$ *and each non-degenerate interval* I, *there is* $x \in I$ *with* $\varphi(x) \neq \lambda x$. *Let* $F(x) = \int_0^x \varphi(t)dt$. *Assume that there exists* $x_0 > 0$ *such that*

$$F(-x_0) = F(x_0) \ .$$

Finally, suppose that F *has no global maxima in* $\mathbf{R}$ *and that*

$$\lim_{|x| \to +\infty} \frac{\varphi(x)}{x} = 0 \ .$$

Then, for each positive and continuous function $\alpha : [0, 1] \to \mathbf{R}$, *there exist* $\lambda^* > 0$ *with the following property: for each continuous function* $g : [0, 1] \times \mathbf{R} \to \mathbf{R}$ *there exists* $\delta > 0$ *such that, for every* $\mu \in [0, \delta]$, *the problem*

$$\begin{cases} -u'' + \alpha(t)u = \lambda^* \alpha(t)\varphi(u) + \mu g(t, u) \quad \textit{in} \quad [0, 1] \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least three classical solutions.

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Received: October 11, 2007; Accepted: November 15, 2007.