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AN APPLICATIONS OF SCHAUDER'S FIXED POINT THEOREM TO BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In general, all results on the existence of solution of the stochastic differential equations are based on the convergence of some approximating sequence by a kind of Picard iteration. The our goal, is to prove existence of solutions of a backward stochastic differential equation with some general assumptions on coefficients functions using the Schauder's fixed point theorem, generalizing some results for the (forward) stochastic differential equations.
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1. Introduction

The linear backward stochastic differential equations (BSDEs) appeared long time ago, both as the equations for the adjoint process in stochastic control, as well as the model behind the Black and Scholes formula for the pricing and hedging of options in mathematical finance. These linear BSDEs with deterministic terminal time was fist solved by Pardoux and Peng (1990, [14]), who introduced the term of the BSDE and proved the existence and uniqueness of adapted solution with Lipschitz coefficients functions. Since then, the interest of the BSDEs, has increased steadily (see [10]), due to the connections of this subject with mathematical finance, stochastic control, and

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partial differential equations. In particular, many efforts have been made to relax the assumptions on the coefficient functions (for instance Mao [11], Lepeltier and San Martin [8], [9], Kobylanski [7], Constantin [5] and Negrea [13]). In general, all results on the existence of solution are based on the convergence of some approximating sequence by a kind of Picard iteration. In this paper, we prove existence of solutions of a backward stochastic differential equation with some general assumptions on coefficients functions using the Schauder's fixed point theorem, generalizing some results for the (forward) stochastic differential equations (see for example [6]).

2. Preliminary results

Throughout this paper $(\Omega, \mathcal{F}, P)$ is a complete probability space endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,1]}$ satisfying the usual hypotheses. We denote with $\mathcal{M}^2(0,1)$ the set of all stochastic processes defined on $\Omega \times [0,1]]$ which are square integrable matingale with respect to the natural filtration $\{\mathcal{F}_t\}$, $0 \leq t \leq 1$.

We consider the following stochastic system

$$\begin{cases} X_t = X_0 + \int_0^t a(s, X(s), Y(s))ds + \int_0^t b(s, X(s), Y(s))dW(s) \\ \int_t^1 f(s, X(s), Y(s))ds + \int_t^1 g(s, X(s), Y(s))dW(s) = X_1 \end{cases} \tag{1}$$

with $a, b, f, g : \Omega \times (0,1) \times \mathbb{R} \times \mathbb{R}$ and the following hypotheses

i) a, b, f and g is $\mathcal{P} \otimes \mathcal{B} \otimes \mathcal{B}$ measurable functions;

ii) $\varphi(\cdot, 0, 0) \in M^2((0,1), \mathbb{R})$, where φ is any functions a, b, f or g;

iii) there exists $u(t)$ a continuous, positive and derivable function on $0 < t \leq 1$ with $u(0) = 0$, having nonnegative derivative $u'(t) \in L([0,1])$, with $u'(t) \to \infty,\ t \to 0^+$ such that

$$|\varphi(t, x_1, y_1) - \varphi(t, x_2, y_2)|^2 \leq \frac{u'(t)}{Ku(t)} \min(|x_1 - x_2|^2, |y_1 - y_2|^2), \tag{2}$$

for all $x_1, x_2, y_1, y_2 \in R,\ 0 \leq t \leq 1$, positive constant K and φ is any function a, b, f or g;

iv) with the same functions $u(t)$ as above,

$$|\varphi(t, x, y)|^2 \leq u'(t) \min(1 + |x|^2, 1 + |y|^2), \tag{3}$$

and X_0 is a finite random variable and X_1 is given $\mathcal{F}_1$-measurable random variable such that $E|X|^2 < \infty$.

Remark. In the above system we have a forward stochastic differential equation (the first equation) and a backward stochastic differential equation (the second equation). In the frame of the stochastic differential equations there exists an important difference from the classical (non-stochastic or deterministic) case: a solution is a stochastic process which satisfies identically the equation and has some common properties but it must to be $\mathcal{F}_t$ measurable for any $0 \leq leq1$. This last properties make a difference between forward and backward stochastic differential equation: in the forward stochastic frame the $\mathcal{F}_t$ measurability is assure using the natural filtration $\mathcal{F}_t = \sigma(X_s,\ 0 \leq s \leq t)$, but in the backward stochastic case (i.e. an equation with a final condition) a natural filtration canot be obtained, so a solution made using the successive approximation method not assure the property of the measurability.

Z.S.Athanassov[1] (1990) proved a uniqueness theorem of Nagumo type for the Cauchy problem generalizing several known uniqueness theorems and sufficient conditions to guarantee the convergence of the Picard successive approximations for the ordinary differential equations. Stochastic generalizations of the results of Athanassov for (forward) stochastic differential equations are given by A. Constantin[2],[3], [4], Gh. Constantin and R. Negrea[6] or R. Negrea[12] and others. But it is known that the uniqueness of the solution of an initial value problem of Cauchy and the convergence of successive approximations are logically independent, i.e. the uniqueness of the solution does not ensure the convergence of successive approximations nor is the converse true.

A very important results on the existence and uniqueness of the solution of the backward stochastic differential equation from the system (1) in the general hypothesis from above is given by the author in [13]

Theorem 2.1. *Let be f and g satisfying the above hypotheses and $X \in L^2(\Omega, \mathcal{F}_1, P, \mathbb{R})$, then there exists a unique pair $(x, y) \in M^2((0,1) \times M^2(0,1), \mathbb{R} \times \mathbb{R})$ which satisfies the equation (1) for $\delta \leq t \leq 1$, for any positive constant δ.*

3. Main result

In this section, we prove that the stochastic (forward-backward) differential system has a solution on some interval $[\delta, 1]$ for any positive constant δ in some general hypotheses of the coefficient functions a, b, f and g as above. This case, with some discontinuity in the initial time moment $t = 0$ is is in according situation of the transition financial markets, where the underlying assets (which is modeling with a forward stochastic differential equation) is over-quoted.

We have the following theorem:

Theorem 3.1. *Let be f and g satisfying the above hypotheses and $X \in L^2(\Omega, \mathcal{F}_1, P, \mathbb{R})$, then there exists a pair $(x, y) \in M^2((0,1), \mathbb{R})$ which satisfy the equation (1) for $\delta \leq t \leq 1$, for any positive constant δ.*

Proof. Let to prove the existence of a solution of system (1) on some interval $[\delta, 1]$ with a positive number δ which will be explained from along to the proof. We will made a reasoning similar to Gh. Constantin and R. Negrea [6].

We consider the operator T defined on $M^2(0,1) \times M^2(0,1)$ by

$$T[x(t), y(t)] = (F[x(t), y(t)], G[x(t), y(t)]) =$$

$$= \begin{pmatrix} x_0 + \int_0^t a(s, x(s), y(s))ds + \int_0^t b(s, x(s), y(s))dW(s) \\ \int_t^1 f(s, x(s), y(s))ds + \int_t^1 g(s, x(s), y(s))dW(s) - X \end{pmatrix}, \ \delta \leq t \leq 1.$$

By hypotheses, these integrals exist and T maps $M^2(0,1) \times M^2(0,1)$ into $M^2(0,1) \times M^2(0,1)$.

In the Banach space $M^2(0,1) \times M^2(0,1)$ we consider the norm

$$|||(x, y)||| = \sqrt{\|x\|2 + \|y\|^2}, \quad \|x\|^2 = E[\sup_{0 \leq t \leq 1} |x(t)|^2].$$

Let $n \in N$ be such that $\sqrt{u(t)} \leq (n-1)q$, $q = \max\{\|x_0\|, \|X\|\}$ and $Q = (2n-1)q\sqrt{2}$. Let $\delta_1 \in (0,1]$ such that $\|x\| \leq Q$ and $\|y\| \leq Q$ and from (3) we have that $\|\phi(t, x, y)\|^2 \leq u'(t)$ for $t \in [b, 1]$ and for every ϕ of the form a, b, f or g.

The set

$$\mathcal{B} = \{(x, y) \in M^2((0,1) \times M^2((0,1) : \ \|x\| \leq Q, \ \|y\| \leq Q \ \text{for } t \in [\delta_1, 1]\}$$

is a closed bounded and convex subset of the Banach space $M^2((0,1) \times M^2(0,1), |||.|||$.

If x and y are in $\mathcal{B}$, we have that

$$\| \int_{\delta_1}^t a(s,x(s),y(s))ds\| \leq \{\int_0^t \|a(s,x(s),y(s))\|^2 ds\}^{1/2} \leq$$

$$\leq \{\int_t^1 u'(s)ds\}^{1/2} = \sqrt{u(t)}$$

and

$$\| \int_{\delta_1}^t b(s,x(s),y(s))dW(s)\| \leq \{\int_0^t \|b(s,x(s),y(s))\|^2 ds\}^{1/2} \leq$$

$$\leq \{\int_t^1 u'(s)ds\}^{1/2} = \sqrt{u(t)},$$

for $\delta_1 \leq t \leq 1$.

For the second equation we have

$$\| \int_t^1 f(s,x(s),y(s))ds\| \leq \{\int_t^1 \|f(s,x(s),y(s))\|^2 ds\}^{1/2} \leq$$

$$\leq \{\int_t^1 u'(s)ds\}^{1/2} \leq \sqrt{u(1)-u(t)} \leq \sqrt{u(t)}$$

and

$$\| \int_t^1 g(s,x(s),y(s))dW(s)\| \leq \{\int_t^1 \|g(s,x(s),y(s))\|^2 ds\}^{1/2} \leq$$

$$\leq \{\int_t^1 u'(s)ds\}^{1/2} \leq \sqrt{u(1)-u(t)} \leq \sqrt{u(t)}.$$

because, from the hypotheses of the function $u(t)$ there exists a positive number δ_2 such that $u(t)-u(1) \leq u(t)$, therefore the above inequalities hold on for any $\delta_2 \leq t \leq 1$

We take $\delta = \max\{\delta_1, \delta_2\}$ and we obtain, for

$$|||T[x,y]||| = \sqrt{\|F[x,y]\|^2 + \|G[x,y]\|^2},$$

the following estimates

$$\|F[x(t),y(t)]\| \leq q + 2(n-1)q = (2n-1)q, \quad \delta \leq t \leq 1,$$

and respectively

$$\|G[x(t),y(t)]\| \leq 2(n-1)q + q = (2n-1)q, \quad \delta \leq t \leq 1$$

which implies

$$|||T[x,y]||| \leq \sqrt{2(2n-1)^2 q^2} = Q, \quad x,y \in \mathcal{B},$$

i.e. $T(\mathcal{B}) \subseteq \mathcal{B}$.

Also, if x, y are in $\mathcal{B}$ we have that

$$\|F[x(t), y(t)] - F[x(s), y(s)]\|^2 \leq u(t) - u(s), \;\; \delta \leq s \leq t \leq 1$$

and respectively

$$\|G[x(t), y(t)] - G[x(s), y(s)]\|^2 \leq u(t) - u(s), \;\; \delta \leq s \leq t \leq 1$$

using similar inequalities as above. Therefore

$$|||T[x(t), y(t)] - T[x(t), y(t)]||| \leq u(t) - u(s), \;\; \delta \leq s \leq t \leq 1$$

and thus the set $T(\mathcal{B})$ is equicontinuous.

Moreover, we have for $x, y, x', y' \in \mathcal{B}$ that

$$\|F[x(t), y(t)] - F[x'(t), y'(t)]\|^2 \leq \int_\delta^t \|a(s, x(s), y(s)) - a(s, x'(s), y'(s))\|^2 ds +$$

$$+ \int_\delta^1 \|b(s, x(s), y(s)) - b(s, x'(s), y'(s))\|^2 ds, \;\; \delta \leq t \leq 1.$$

and respectively

$$\|G[x(t), y(t)] - G[x'(t), y'(t)]\|^2 \leq \int_t^1 \|f(s, x(s), y(s)) - f(s, x'(s), y'(s))\|^2 ds +$$

$$\int_t^1 \|g(s, x(s), y(s)) - g(s, x'(s), y'(s))\|^2 ds \;\; \delta \leq t \leq 1.$$

It is easy to see that for every ϕ of the form a or b we have

$$\int_\delta^t \|\phi(s, x(s), y(s)) - \phi(s, x'(s), y'(s))\|^2 ds \leq$$

$$\leq \int_\delta^t \frac{u(s)}{Ku'(s)} (\min\{|x(s) - x'(s)|, |y(s) - y'(s)|\}) ds, \;\; \delta \leq t \leq 1$$

and using the Lebesgue's convergence theorem results that F is continuous function. Analogous, we obtain It is easy to see that for every ψ of the form f or g we have

$$\int_t^1 \|\psi(s, x(s), y(s)) - \psi(s, x'(s), y'(s))\|^2 ds \leq$$

$$\leq \int_t^1 \frac{u(s)}{Ku'(s)} (\min\{|x(s) - x'(s)|, |y(s) - y'(s)|\}) ds, \;\; \delta \leq t \leq 1$$

and similar, from the Lebesgue's convergence theorem, G is a continuous function, and then T is continuous.

Applying the Schauder's fixed point theorem we obtain that T has a fixed point $\mathcal{B}$, thus the stochastic differential system (1) has a solution on $[\delta, 1]$.

Using similar arguments as in [15] or [13] we have that there exists a solution on $[\delta, 1]$ for any positive number δ. $\square$

References

[1] Z. S. Athanassov, *Uniqueness and convergence of successive approximations for ordinary differential equations*, Math. Japonica, **53**(1990), 351-367.

[2] A. Constantin, *Global Existence of Solutions for Perturbed Differential Equations*, Annali di Mat.Pura ed Appl., Serie IV, **CLXVIII**(1995), 237-299.

[3] A. Constantin, *On the Existence and Pathwise Uniqueness of Solutions of Stochastic Differential Equations*, Stochastic and Stochastic Reports, **56**(1996), 227-239.

[4] A. Constantin, *On the Existence and Uniqueness of Solutions of McShane Type Stochastic Differential Equations*, Stoch. Anal. Appl., **16**(1998), 217-229.

[5] Gh. Constantin, *The uniqueness of solutions of perturbed backward stochastic differential equations*, J. Math. Anal. Appl., **300**(2004), 12-16.

[6] Gh. Constantin, R. Negrea, *An application of Schauder's Fixed Point Theorem in Stochastic McShane Modeling*, Fixed Point Theory, **5**(2004), 37-52.

[7] M. Kobylanski, *Backwrad stochastic differential equations and partial differential equations with quadratic growth*, The Annals of Probability, **28**(2000), 558-602.

[8] J.-P. Lepeltier, J. San Martin, *Existence for BSDE with sperlinear-quadratic coefficient*, Stoch. Stoch. Reports, **63**(1998), 227-240.

[9] J.-P. Lepeltier, J. San Martin, *On the existence or non-existence of solutions for certain backward stochastic differential equations*, Bernoulli, **8**(2002), 123-137.

[10] J. Ma, P. Protter, J. Yong, *Solving Forward-Backward Differential Equation Explicitly - A Four Step Scheme*, Probab. Theory Relat. Fields, **98**(1994), 339-359.

[11] X. Mao, *Adapted Solutions of Backward Stochastic Differential Equations with non Lipschitz coefficients*, Stoch. Proc. and Their Appl., **58**(1995), 281-292.

[12] R. Negrea, *On the uniqueness and convergence of successive approximations for a class of stochastic differential equations*, Analele Univ. Bucuresti, Ser. Mat., **52**(2003), 225-234.

[13] R. Negrea, *On the Existence and Uniqueness of Solutions for certain Backward Stochastic Differential Equations* (2006), to appear.

[14] E. Pardoux, S. G. Peng, *Adapted solution of a backward stochastic differential equation*, Systems & Control Letters, **14**(1990), 55-61.

[15] G. Vidossich, *Global convergence of successive approximations*, J. Math. Anal. Appl., **45**(1974), 285-292.

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