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NEW RESULTS ABOUT SOME NONLINEAR OPERATORS

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Abstract. We present in this paper three new classes of nonlinear operators related to the study of complementarity problems and variational inequalities. Some open subjects related to these operators are also put in evidence.
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1. Introduction

In some of our recent papers [10], [11], [13]-[15], [21] and [22] we put in evidence an interaction between the nonlinear analysis and the complementarity theory, i.e., the nonlinear analysis has interesting applications to the study of complementarity theory and conversely, the complementarity theory put to nonlinear analysis interesting problems. The goal of complementarity theory is the study of a class of mathematical models used in optimization, game theory, economics, mechanics, elasticity, engineering and robotics among others [9]-[11], [13], [14], [19], [21].

Generally the nonlinear analysis is used as a mathematical tool in the study of solvability of functional equations. The relations of nonlinear analysis with

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complementarity theory offer to nonlinear analysis other kinds of applications. Another aspect of relations between nonlinear analysis and complementarity theory is the fact that recently we arrived to put in evidence three classes of new nonlinear operators in infinite dimensional vector spaces, i.e., B-quasi-bounded operators, scalarly compact operators and normal operators associated to variational inequalities (in particular to complementarity problems).

We present these nonlinear operators because we consider that their study may be new subjects in nonlinear analysis.

The results related to these operators are selected only to support the importance of each class of operators. The proofs will be presented in some of our papers, now in preparation.

The operators considered in this paper have interesting applications to complementarity theory and to the study of variational inequalities.

Some open subjects related to these operators will be also put in evidence.

2. Preliminaries

We denote by $(E, \|\cdot|)$ a Banach space and by $(H, \langle\cdot\rangle)$ a Hilbert space.

A closed pointed convex cone in E or in H is a subset $\mathbb{K}$ of E or of H satisfying the following properties:

1) $\mathbb{K}+\mathbb{K} \subseteq \mathbb{K}$,

2) $\lambda\mathbb{K} \subseteq \mathbb{K}$ for any $\lambda \in \mathbb{R}_+$,

3) $\mathbb{K} \cap (-\mathbb{K}) = \{0\}$,

4) $\mathbb{K}$ is a closed subset (in the topological sense).

Any convex cone considered in this paper will be closed and pointed.

If E^* is the topological dual of E we denote by $\langle E, E^*\rangle$ a duality (pairing) between E and E^* defined by a separating bilinear mapping $\langle\cdot,\cdot\rangle : E \times E^* \to \mathbb{R}$. If $\mathbb{K} \subset E$ is a pointed convex cone, the dual cone of $\mathbb{K}$ is by definition:

$$\mathbb{K}^* = \{y \in E^* \mid \langle x, y\rangle \geq 0 \text{ for any } x \in \mathbb{K}\}.$$

If $\mathbb{K}$ is a closed convex cone in a Hilbert space H then the projection operator onto $\mathbb{K}$, denoted by $P_{\mathbb{K}}(x)$ is the unique element in $\mathbb{K}$ such that

$$\|x - P_{\mathbb{K}}(x)\| \leq \|x - y\|, \text{ for any } y \in \mathbb{K}.$$

For other properties of $P_{\mathbb{K}}$, the reader is referred to [10].

Let $(E, \|\cdot\|)$ be a Banach space, E^* the topological dual of E and $\langle E, E^*\rangle$ a duality.

If $\mathbb{K} \subset E$ is a closed convex cone and $f : E \to E^*$ a mapping, the nonlinear complementarity problem defined by $\mathbb{K}$ and f is:

$$NCP(f, \mathbb{K}) : \quad \begin{cases} \text{find } x_* \in \mathbb{K} \text{ such that} \\ f(x_*) \in \mathbb{K}^* \text{ and } \langle x_*, f(x_*)\rangle = 0. \end{cases}$$

If $D \subset E$ is closed convex set, the variational inequality (in Hartman-Stampacchia's sense) is:

$$VI(f, D) : \quad \begin{cases} \text{find } x_* \in D \text{ such that} \\ \langle x - x_*, f(x_*)\rangle \geq 0, \text{ for any } x \in D. \end{cases}$$

If D is a closed convex cone then in this case we can prove that, the problem $VI(f, D)$ is exactly the problem $NCP(f, D)$.

We recall that a mapping $f : E \to E$ (or $f : E \to E^*$) is completely continuous if f is continuous and for any bounded set $B \subset E$, $\overline{f(B)}$ is a compact set.

Also, we say that f is demicontinuous on a subset $D \subset E$ if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ strongly convergent to an element x^* we have that $\{f(x_n)\}_{n\in\mathbb{N}}$ is weakly convergent to $f(x_*)$.

A mapping $f : E \to E^*$ is monotone (in Kachurovskii-Minty-Browder's sense) if $\langle x - y, f(x) - f(y)\rangle \geq 0$ for any $x, y \in E$ and f is anti-monotone if $\langle x - y, f(x) - f(y)\rangle \leq 0$ for any $x, y \in E$.

Let $(E, \|\cdot|)$ be a Banach space, $\mathbb{K} \subset E$ a closed convex cone and $f : E \to E$ a mapping.

We say that f has an asymptotic derivative along $\mathbb{K}$ if there exists a linear and continuous mapping $T : E \to E$ (i.e., $T \in \mathcal{L}(E, E)$) such that

$$\lim_{\substack{\|x\|\to+\infty \\ x\in\mathbb{K}}} \frac{\|f(x) - T(x)\|}{\|x\|} = 0.$$

If $\mathbb{K}$ is a generating cone i.e., $E = \mathbb{K} - \mathbb{K}$ then in this case T is unique and it is denoted by $f^\infty_{\mathbb{K}}$. Obviously, $\mathbb{K}$ can be E.

The notion of asymptotic derivative is due to M. A. Krasnoselskii and the reader can find references on this subject in [12], [21] and [22].

We note that the asymptotic derivative is a fundamental mathematical tool in nonlinear analysis.

Finally, we recall the notion of semi-inner-product.

Let $(E, \|\cdot\|)$ be a Banach space. A semi-inner-product (in Lumer's sense) is a mapping $[\cdot,\cdot] : E \times E \to \mathbb{R}$ satisfying the following properties:

(s_1) $[x+y, z] = [x, z] + [y, z]$, for all $x, y, z \in E$,

(s_2) $[\lambda x, y] = \lambda[x, y]$, for all $x, y \in E$ and any $\lambda \in \mathbb{R}$,

(s_3) $[x, x] > 0$, for any $x \in E \setminus \{0\}$,

(s_4) $|[x, y]|^2 \leq [x, x][y, y]$, for all $x, y \in E$.

It is known that any Banach space can be endowed with a semi-inner-product.

A semi-inner-product defines a norm on E by $\|x\|_s = [x, x]^{1/2}$.

It is possible to define on E a semi-inner-product such that, $[x, x] = \|x\|^2$. In this case we say that the semi-inner-product is compatible with the norm $\|\cdot\|$ given on E. A semi-inner-product in Deimling's sense is a mapping $[\cdot,\cdot]_d : E \times E \to \mathbb{R}$ defined by:

$$[x, y]_d = \|y\| \lim_{t \to 0_+} \frac{\|y + tx\| - \|y\|}{t}, \text{ for any } x, y \in E.$$

We note that this semi-inner-product is only subadditive in the first variable and $[lx, y] = l[x, y]$, for any $l > 0$. For more information about semi-inner-products the reader is referred to the references cited in [15], [21], [22].

3. B-quasi-bounded operators

First, we recall the notion of quasi-bounded operator, defined in 1962 by A Granas [7]. This notion was defined as a mathematical tool for the fixed point theory.

Let $(E, \|\cdot\|)$ be a Banach space and $f : E \to E$ a mapping.

Definition 3.1. [7] We say that f is quasi-bounded if and only if

$$\limsup_{\|x\| \to +\infty} \frac{\|f(x)\|}{\|x\|} < +\infty.$$

If f is quasi-bounded we denote

$$[f]_{qb} = \limsup_{\|x\| \to +\infty} \frac{\|f(x)\|}{\|x\|} = \inf_{r>0} \sup_{\|x\| \geq r} \frac{\|f(x)\|}{\|x\|},$$

and we say that $[f]_{qb}$ is the quasi-norm of f.

The notion of quasi-bounded operator has been used by several authors in the study of several problems related to the study of fixed-points, to the study of surjectivity of nonlinear operators and to the study of other problems considered in nonlinear analysis or in applied mathematics [5], [7], [14], [15], [24, [27], [28], [35].

In a few of our recent papers we presented some applications of quasi-bounded operators to the study of complementarity problems, [14], [15], [21].

We note also the fact that if $f : E \to E$ has an asymptotic derivative $T \in \mathcal{L}(E, E)$, then f is quasi-bounded and $[f]_{qb} = \|T\|$.

This fact implies that many integral operators (as Hammerstein or Urysohn) are quasi-bounded, because these operators, under some conditions have an asymptotic derivative.

Now, we present a generalization of the notion of quasi-bounded operators.

Let $(E, \|\cdot\|)$ be a Banach space and $B : E \times E \to \mathbb{R}$ a mapping satisfying the following properties:

(b_1) $B(\lambda x, y) = \lambda B(x, y)$, for any $x, y \in E$ and any $\lambda \in \mathbb{R}_+ \setminus \{0\}$,

(b_2) $B(x, x) > 0$, for any $x \in E \setminus \{0\}$.

Examples

1. If E is a Hilbert space and $\langle\cdot,\cdot\rangle$ is the inner-product defined on E, then in this case we can take $B(\cdot,\cdot) = \langle\cdot,\cdot\rangle$.

2. Let $(E, \|\cdot\|)$ be an arbitrary Banach space and let $[\cdot,\cdot]$ be a semi-inner-product in Lumer's sense or in Deimling sense. In this case we can take $B(\cdot,\cdot) = [\cdot,\cdot]$.

3. If $(E, \|\cdot\|)$ is a Banach space and ($\mathcal{B} : E \times E \to \mathbb{R}$ is a bilinear form which is coercive, i.e., there is a constant $K > 0$ such that $\mathcal{B}(x, x) \geq k\|x\|^2$, for any $x \in E$, then in this case we can take $B(\cdot,\cdot) = \mathcal{B}(\cdot,\cdot)$.

4. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $E = C([0, 1], H)$ the normed vector space of continuous functions from $[0, 1]$ into H with the norm $\|x\| = \sup_{t\in[0,1]} \|x(t)\|_H$, where $\|\cdot\|_H$ is the norm of the space H. In this case we can take on the space E the mapping B defined by

$$B(x, y) = \sup_{t\in[0,1]} \langle x(t), y(t)\rangle,$$

or

$$B(x, y) = \int_0^1 \langle x(t), y(t)\rangle dt.$$

Let $(E, \|\cdot\|)$ be a Banach space and $B : E \times E \to \mathbb{R}$ a mapping satisfying the properties (b_1) and (b_2).

Definition 3.2. [13] We say that a mapping $f : E \to E$ is B-quasi-bounded with respect to a closed convex cone $\mathbb{K} \subset E$ if and only if

$$\limsup_{\substack{\|x\| \to +\infty \\ x \in \mathbb{K}}} \frac{B(f(x), x)}{B(x, x)} < +\infty.$$

Remark. In Definition 3.2, the cone $\mathbb{K}$ can be the space E.

If f is B-quasi-bounded we denote

$$[f]^{\mathbb{K}}_{B_{qb}} = \limsup_{\substack{\|x\| \to +\infty \\ x \in \mathbb{K}}} \frac{B(f(x), x)}{B(x, x)}.$$

Remark. If in Definition 3.2, we have $\mathbb{K} = E$, we denote in this case $[f]^{\mathbb{K}}_{B_{qb}}$ by $[f]_{B_{qb}}$.

If E is a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and B is the inner product $\langle\cdot,\cdot\rangle$ then in this case we say that $f : H \to H$ is scalarly-quasi-bounded (shortly S-quasi-bounded) if f is $\langle\cdot,\cdot\rangle$-quasi-bounded with respect to $\mathbb{K}$ (or with respect to H) in the sense of Definition 3.2.

It is easy to observe that on a Hilbert space, taking $B = \langle\cdot,\cdot\rangle$, we have that any quasi-bounded mapping $f : H \to H$ (in Granas' sense) is S-quasi-bounded.

Also, if B is a semi-inner-product in Lumer's sense on a Banach space $(E, \|\cdot\|)$, then any quasi-bounded mapping in Granas' sense is B-quasi-bounded if the semi-inner-product is compatible with the norm of E.

This result is also true if B is a bilinear functional satisfying the following properties:

(i) $|B(x, y)| \leq M\|x\|\|y\|$, for some $M > 0$,

(ii) $B(x, x) \geq \rho\|x\|^2$, for some $\rho > 0$.

Now, we give an interesting fixed point theorem for B-quasi-bounded mappings. Our theorem is with respect to a closed convex cone $K \subset E$ but it is valid on the space E.

Let $(E, \|\cdot\|)$ be a Banach space and $\mathbb{K} \subset E$ a closed convex cone (not necessarily pointed). If $\Omega \subset \mathbb{K}$ is a non-empty subset, we denote by $\overline{\Omega}$, $\partial\Omega$ and $conv(\Omega)$ the closure, the boundary and the convex hull of Ω in $\mathbb{K}$. Let $\mathcal{P}_b(\mathbb{K})$ the collection of bounded subset of $\mathbb{K}$.

We recall the notion of measure of noncompactness. We say that a function $\alpha : \mathcal{P}_b(\mathbb{K}) \to [0, +\infty[$ is a measure of noncompactness on $\mathbb{K}$ if the following properties are satisfied:

(α_1) $\alpha(A) = 0$ if and only if $\overline{A}$ is compact,

(α_2) $\alpha(A) = \alpha(\overline{A})$,

(α_3) $A_1 \subseteq A_2$ implies $\alpha(A_1) \leq \alpha(A_2)$,

(α_4) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,

(α_5) $\alpha(\lambda A) = \lambda\alpha(A)$, for any $\lambda \in \mathbb{R}_+$,

(α_6) $\alpha(conv(A)) = \alpha(A)$,

(α_7) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

We cite as example of a measure of noncompactness the Kuratowski measure of noncompactness defined by:

$$\alpha(A) = \inf\{r > 0 \mid A \text{ admits a finite cover by sets of diameter at most } r\}$$

There exist several papers and books devoted to measures of noncompactness as for example [1]-[3].

Definition 3.3. [37] We say that a continuous mapping $f : \Omega \to \mathbb{K}$ is a countable α-condensing mapping if $\alpha(f(D)) < \alpha(D)$, for each countable bounded set $D \subset \Omega$ with $\alpha(D) > 0$.

Remark. The fact that in this notion is considered only countable bounded sets is important in some applications with differential and integral operators of vector functions in nonseparable Banach spaces [8], [6], [37], [25], [26].

The next definition is also necessary.

We say that a continuous homotopy $h : [0, 1] \times \Omega \to \mathbb{K}$ is countable α-condensing if $\alpha(h([0, 1] \times D)) < \alpha(D)$ for each countable bounded set $D \subseteq \Omega$ with $\alpha(D) > 0$.

We note that in 1999, M. Väth defined a topological fixed-point index for countable α-condensing mapping [37].

Let $\Omega \subset \mathbb{K}$ be a non-empty set and $f : \overline{\Omega} \to \mathbb{K}$ a countable α-condensing mapping without fixed points on $\partial\Omega$. In this case the fixed-point index denoted by $ind_{\mathbb{K}}(f, \Omega)$ is well defined and it has the following properties.

Proposition 3.1. [37] *Let Ω be a non-empty bounded open set in $\mathbb{K}$ and $f : \overline{\Omega} \to \mathbb{K}$ a countable α-condensing mapping such that f has no fixed-point on $\partial\Omega$. Then the following properties of $ind_{\mathbb{K}}$ are satisfied:*

(1) Existence: if $ind_{\mathbb{K}}(f, \Omega) \neq 0$ then f has a fixed-point in Ω,

(2) Normalization: if $f \equiv 0$ and $0 \in \Omega$ then $ind_{\mathbb{K}}(f, \Omega) = 1$,

(3) Homotopy invariance: if $h : [0,1] \times \overline{\Omega} \to \mathbb{K}$ is a countable α-condensing homotopy such that $h(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial\Omega$ then $ind_{\mathbb{K}}(h(0,\cdot), \Omega) = ind_{\mathbb{K}}(h(1,\cdot), \Omega)$.

Proof. For the proof of this proposition the reader is referred to [37]. □

Using the topological fixed-point index for countable α-condensing mappings and its properties given in Proposition 3.1, we proved the following interesting fixed-point theorem for B-quasi-bounded mapping.

Theorem 3.2. *Let $(E, \|\cdot\|)$ be a Banach space, $\mathbb{K} \subset E$ a closed convex cone and $f : E \to E$ a mapping. If the following assumptions are satisfied:*

(i) f is countable α-condensing,

(ii) $f(\mathbb{K}) \subseteq \mathbb{K}$,

(iii) f is B-quasi-bounded and $[f]^{\mathbb{K}}_{B_{qb}} < 1$,

then f has a fixed point in $\mathbb{K}$.

Proof. The proof of this theorem is given in [14]. □

Open subjects

(1) It is interesting to find surjectivity theorem for B-quasi-bounded operators.

(2) It is known that the notion of quasi-bounded operators (in Granas' sense) can be extended to set-valued mappings [28]. Because this fact, it is interesting to extend the notion of B-quasi-bounded operators to set-valued mappings.

We propose the following extension.

Let $(E, \|\cdot\|)$ be a Banach space and $f : E \to E$ a set-valued mapping. We define:

$$\varphi(B(f(x), x)) := \sup\{\|y\| \mid y \in B(v, x),\ v \in f(x)\}.$$

If

$$\limsup_{\|x\| \to +\infty} \frac{\varphi(B(f(x), x))}{B(x, x)} < +\infty,$$

we say that the set-valued mapping f is B-quasi-bounded.

(3) It is interesting to study this class of set-valued mappings.

(4) It is interesting to study the eigenvalues of B-quasi-bounded mappings.

4. Scalarly compact operators

We present in this section the class of nonlinear scalarly compact operators. The origin of this notion is property $(S)_+$, defined in 1968 by F. E. Browder as a mathematical tool in nonlinear analysis, to replace the compactness when this is not present [4].

Definition 4.1. [Browder] Let $(E, \|\cdot\|)$ be a Banach space, E^* the topological dual of E and $\langle\cdot,\cdot\rangle$ a duality between E and E^*. A mapping $f : E \to E^*$ is said to satisfy condition $(S)_+$ if for any sequence $\{x_n\}_{m\in\mathbb{N}} \subset E$ weakly convergent to an element x^* and such that $\limsup_{n\to\infty}\langle x_n - x_*, T(x_n)\rangle \leq 0$ we have that $\{x_n\}_{n\in\mathbb{N}}$ is norm convergent to x^*.

There exist many examples of operators satisfying condition $(S)_+$ [4], [9], [10], [16], [20]. In particular any strongly monotone operator satisfies condition $(S)_+$. We note also that under particular conditions some partial differential operators satisfy condition $(S)_+$

Let $(E, \|\cdot\|)$ be a Banach space, $D \subset E$ a closed convex subset. The following notion is due to G. Isac.

Definition 4.2. [15], 16] We say that a mapping $f : D \to E^*$ is scalarly compact if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ weakly convergent to an element $x^* \in D$, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of the sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\limsup\limits_{k\to\infty}\langle x_{n_k} - x_*, f(x_{n_k})\rangle \leq 0$.

Examples

(1) If $f : E \to E^*$ is completely continuous then f is scalarly compact.

(2) If $h : E \to E^*$ is completely continuous, $g : E \to E^*$ is monotone, then we can prove that, $f(x) = h(x) - g(x)$, for any $x \in E$ is scalarly compact.

(3) If E is a Banach space and $J : E \to E^*$ is a duality mapping such that for any $x \in E$, $J(x)$ is a singleton, then for any completely continuous mapping $h : E \to E^*$, the mapping $f(x) = h(x) - J(x)$, for any $x \in E$ is scalarly compact.

(4) If $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space and $h : H \to H$ is a completely continuous mapping, then the mapping $f(x) = h(x) - x$ is not completely continuous but it is scalarly compact.

(5) If $f : E \to E^*$ is antimonotone (i.e., $-f$ is monotone), then f is scalarly compact.

(6) If $h : E \to E^*$ is scalarly compact and $g : E \to E^*$ is completely continuous then, $f(x) = h(x) + g(x)$, for any $x \in E$ is scalarly compact.

(7) If h and g are scalarly compact mappings from E into E^* then, for any scalars $a, b > 0$, the mapping $f(x) = ah(x) + bg(x)$ is scalarly compact.

(8) If for $f : E \to E^*$ there exists a completely continuous mapping $h : E \to E^*$ such that $\langle y, f(x)\rangle \leq |\langle y, h(x)\rangle|$ for any $x, y \in D$ (where D is a closed convex set) then f is scalarly compact.

Let $(E, \|\cdot\|)$ be a Banach space, E^* the topological dual of E and $D \subset E$ a nonempty subset.

Definition 4.3. [9], [16], [20]. We say that a mapping $f : D \to E^*$ satisfies condition $(S)^1_+$ if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ with $(w) - \lim_{n\to\infty} x_n = x_*$, $(w) - \lim_{n\to\infty} f(x_n) = u \in E^*$ and $\lim_{n\to\infty} \langle x_n, f(x_n)\rangle \leq \langle x_*, u\rangle$, we have that $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence norm convergent to x_*.

Remark. We introduced condition $(S)^1_+$ in [9] and we studied it in a joint paper with S. M. Gowda [20]. This condition has been considered and used by several authors.

We note that any mapping satisfying condition $(S)_+$ satisfies condition $(S)^1_+$ too.

Let $(E, \|\cdot\|)$ be a Banach space, E^* the topological dual of E and $\mathcal{L}(E, E^*)$ the Banach space of linear continuous operators from E into E^*. Let $K \subset E$ be an unbounded closed convex set. The following notion is due to G. Isac.

Definition 4.4. We say that $T \in \mathcal{L}(E, E^*)$ is a scalar asymptotic derivative of a mapping $f : E \to E^*$ along the set $\mathbb{K}$ if

$$\limsup_{\substack{x\in\mathbb{K}\\ \|x\|\to\infty}} \frac{\langle x, f(x) - T(x)\rangle}{\|x\|^2} \leq 0.$$

We denote T by f^∞_s.

Remark. If is easy to show that if f has an asymptotic derivative T, along $\mathbb{K}$ then, T is a scalar asymptotic derivative of F along the same set $\mathbb{K}$.

Now, we cite an existence result for variational inequalities and complementarity problems.

Theorem 4.1. *Let $(E, \|\cdot\|)$ be a reflexive Banach space and $T_1, T_2 : E \to E^*$ demicontinuous mappings.*

If the following assumptions are satisfied:

(1) T_1 is bounded (i.e., $D \subset E$, D bounded implies $f(D)$ bounded) and satisfies condition $(S)^1_+$,

(2) T_2 is scalarly compact.

Then for every non-empty bounded closed convex set $D \subset E$, the problem $VI(T_1 - T_2, D)$ has a solution.

Proof. The proof is based on several technical details. The scalarly compactness and condition $(S)^1_+$ are essential in this proof. □

As application of Theorem 4.1 we have the following existence theorem for variational inequalities with respect to unbounded closed convex sets.

Theorem 4.2. *Let $(E, \|\cdot\|)$ be a reflexive Banach space, $\mathbb{K} \subset E$ an unbounded closed convex set such that $0 \in \mathbb{K}$.*

Let $T_1, T_2 : E \to E^*$ be two demicontinuous mappings. If the following assumptions are satisfied:

(1) T_1 is bounded and satisfies condition $(S)^1_+$,

(2) T_2 is scalarly compact,

(3) There exist $r > 0$ and $c > 0$ such that $c\|x\|^2 \leq \langle x, T_1(x)\rangle$ for all $x \in \mathbb{K}$ with $\|x\| > r$,

(4) T_2 has a scalar asymptotic derivative $T^\infty_{2,s}$ such that $\|T^\infty_{2,s}\| < c$,

then the problem $VI(T_1 - T_2, \mathbb{K})$ has a solution.

In particular, if $\mathbb{K}$ is a closed pointed convex cone, then the problem $NCP(T_1 - T_2, \mathbb{K})$ has a solution.

Proof. The proof of this result is given in [16]. □

Definition 4.5. [16] Let T_1, T_2 be two mappings from E into E^*. We say that T_1, T_2 satisfy condition (C) if there exists $r > 0$ such that

$$\inf\{\langle x, T_1(x)\rangle \mid x \in \mathbb{K},\ \|x\| = r\} = \rho_1 > 0$$

and

$$\inf\{\langle x, T_2(x)\rangle \mid x \in \mathbb{K},\ \|x\| = r\} = \rho_2 > 0.$$

The following result has interesting applications to the study of solvability of complementarity problems depending of parameters.

Theorem 4.3. *Let $(E, \|\cdot\|)$ be a reflexive Banach space, $\mathbb{K} \subset E$ a closed pointed convex cone and $T_1, T_2 : E \to E^*$ two demicontinuous mappings. If the following assumptions are satisfied:*

(1) T_1 is bounded and satisfies condition $(S)^1_+$,

(2) T_2 is scalarly compact on $\mathbb{K}$,

(3) T_1, T_2 satisfy condition (C).

Then for any λ such that, $0 < \lambda < \dfrac{\rho_1}{\rho_2}$ the problem $NCP(T_1 - \lambda T_2, \mathbb{K})$ has a solution. This solution is a nontrivial solution if $T_1(0) - \lambda T_2(0) \notin \mathbb{K}^*$.

Proof. The proof of this result is given in [16]. □

Open subjects

(1) It is useful to find new classes of scalarly compact operators.

(2) It is interesting to find new applications of scalarly compact operators to variational inequalities or complementarity problems defined by differential or integral operators.

5. Normal operators for variational inequalities and complementarity problems in infinite dimensional spaces

Let $(\mathbb{R}_n, \langle \cdot, \langle)$ be the n-dimensional Euclidean space $D \subset \mathbb{R}^n$ a non-empty closed convex set and $f : \mathbb{R}^n \to \mathbb{R}^n$ a mapping. Let P_D be the projection onto D. The normal operator defined by the set D and the mapping f is:

$$\mathcal{N}(D, f)(x) = f(P_D(x)) + x - P_D(x), \text{ for any } x \in \mathbb{R}^n.$$

This operator has been studied until now in $\mathbb{R}^n$ by several authors and especially by S. M. Robinson [29]-[33].

The importance of the normal operator $\mathcal{N}(D, f)$ is supported by the fact that we can solve a variational inequality or a complementarity problem (which are variational problems) by solving an equation of the form

$$\mathcal{N}(D, f)(x) = 0.$$

Several remarkable properties of the operator $\mathcal{N}(D, f)$ were established in [29]-[33] and in other papers.

We note that when D is a polyhedral set, the operator $\mathcal{N}(D, f)$ is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

We consider that it is interesting to study the operator $\mathcal{N}(D, f)$ in an infinite dimensional Hilbert space and to investigate the possibility to extend this operator to some classes of Banach spaces.

Remark. In infinite dimensional Hilbert spaces the operator $\mathcal{N}(D, f)$ can not be completely continuous and generally it is not directionally derivable.

This section may be considered as a starting point of a study of the normal operator in infinite dimensional spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\Omega \subset H$ a closed convex set (generally the set Ω is supposed to be also unbounded). We denote by P_Ω the projection operator onto Ω. Let $f : H \to H$ be a continuous mapping.

Definition 5.1. The normal operator associated to Ω and f is:

$$\mathcal{N}(\Omega, f)(x) = f(P_\Omega(x)) + (x - P_\Omega(x)), \text{ for any } x \in H.$$

We recall the following classical result.

Proposition 5.1. *Let x^* and z^* be two elements in H. We have $x_* = P_\Omega(z_*)$ if and only if $\langle z_* - x_*, x_* - x\rangle \geq 0$ for any $x \in \Omega$.*

Proof. A proof of this result is in [10]. □

Using Proposition 5.1 we can prove the following result.

Theorem 5.2. *If $\mathcal{N}(D, f)(z_*) = 0$ then $x_* = P_\Omega(z_*)$ is a solution to the following variational inequality:*

$$VI(\Omega, f): \quad \begin{cases} \textit{find } x_* \in \Omega \textit{ such that} \\ \langle f(x_*), x - x_*\rangle \geq 0 \textit{ for any } x \in \Omega. \end{cases}$$

Proof. A proof of this result is in [10]. □

A consequence of Theorem 5.2 is the fact that the solvability of the variational problem $VI(\Omega, f)$ is reduced to the solvability of equation $\mathcal{N}(\Omega, f)(x) = 0$.

We note that in many solvability theorems for nonlinear equations are used or the monotonicity or the complete continuity.

Because this fact we will define two new variants of the normal operator, such that one will be completely continuous, and another will be monotone.

Let $\varphi : H \to H$ be an arbitrary completely continuous mapping.

Definition 5.2. The normal operator associated to Ω, f and φ is:

$$\mathcal{N}_\varphi(\Omega, f)(x) = f(P_\Omega(\varphi(x))) + \varphi(x) - P_\Omega(\varphi(x)), \text{ for any } x \in H.$$

We can prove the following result.

Theorem 5.3. *If $\mathcal{N}_\varphi(\Omega, f)(x_*) = 0$, then the element $u_* = P_\Omega(\varphi(x_*))$ is a solution to the following variational inequality:*

$$VI(\Omega, f): \quad \begin{cases} \textit{find } u_* \in \Omega \textit{ such that} \\ \langle f(u_*), u - u_*\rangle \geq 0 \textit{ for any } u \in \Omega. \end{cases}$$

Remark. The operator $\mathcal{N}_\varphi(\Omega, f)$ is completely continuous, if f continuous.

Now we indicate two particular examples of mappings which can be used as mapping φ, in Definition 5.2.

(i) If $\mathbb{K}_0 \subset H$ is an arbitrary locally compact convex cone (i.e., $\mathbb{K}_0$ has a compact base), then we can take in Definition 5.2 $\varphi = P_{\mathbb{K}_0}$.

(ii) If $H = L^2(D, \mu)$, then in this case we can take as mapping φ any completely continuous integral operator (linear or nonlinear).

Let α be a strictly positive real number.

Definition 5.3. The normal operator associated to Ω, f and α is:

$$\mathcal{N}_\alpha(\Omega, f)(x) = f(P_\Omega(x)) + \alpha(x - P_\Omega(x)), \text{ for any } x \in H.$$

We can prove the following result.

Theorem 5.4. *If $\mathcal{N}_\alpha(\Omega, f)(u_*) = 0$, then the element $x_* = P_\Omega(u_*)$ is a solution to the variational inequality:*

$$VI(\Omega f) : \quad \begin{cases} \textit{find } x_* \in \Omega \textit{ such that} \\ laf(x_*), x - x_*\rangle \geq 0 \textit{ for any } x \in \Omega. \end{cases}$$

Now, we give a condition about α which implies that $\mathcal{N}_\alpha(\Omega, f)$ is monotone operator.

We say that a mapping $f : H \to H$ is cocoercive with modulus $\beta > 0$ on the set $\Omega \subset H$ if

$$\langle x - y, f(x) - f(y)\rangle \geq \beta \|f(x) - f(y)\|^2, \text{ for any } x, y \in \Omega.$$

Also, we recall that $f : H \to H$ is strongly monotone with modulus $\rho > 0$ if

$$\langle x - y, f(x) - f(y)\rangle \geq \rho\|x - y\|^2, \text{ for all } x, y \in \Omega.$$

We note that the notion of cocoercive mapping has been used in several papers related to numerical methods for variational inequalities with monotone operators.

We have the following result.

Theorem 5.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\Omega \subset H$ a closed convex set and $f : H \to H$ a mapping.*

(i) If f is cocoercive with modulus $\beta > 0$ on Ω, then for any $\alpha > \dfrac{1}{4\beta}$, the operator $\mathcal{N}_\alpha(\Omega, f)$ is monotone on Ω.

(ii) If f is strongly monotone with modulus $\rho > 0$ and Lipschitzian with modulus $\delta > 0$ on Ω, then for any $\alpha > \dfrac{\delta^2}{4\rho}$ the operator $\mathcal{N}_\alpha(\Omega, f)$ is strongly monotone on Ω.

Proof. The proof is based on several technical details and will be given in one of my future paper. □

Now, we give two solvability theorems applicable to the nonlinear equations:

(a) $\mathcal{N}_\alpha(\Omega, f)(x) = 0$,

(b) $\mathcal{N}_\varphi(\Omega, f)(x) = 0$.

First, we recall the following notion.

Definition 5.4. [23] We say that a mapping $f : E \to E^*$ is hemicontinuous on a set $\Omega \subset E$ if for any $v, x \in E$, $u \in \Omega$ and a real t such that, $u + tv \in \Omega$ we have:

$$\lim_{t\to 0}\langle x, f(u + tv)\rangle = 0.$$

(In this definition E is a Banach space, E^* is the topological dual of E and $\langle \cdot, \cdot \rangle$ is a duality between E and E^*.)

Theorem 5.6. [Kachurovskii] *Let E be a reflexive Banach space and $D \subset E$ a bounded closed convex set such that $0 \in Int(D)$.*

Let $B(0, r)$ be an open ball such that $D \subset B(0, r)$ and let $f : B(0, r) \to E^*$ be a mapping satisfying the following conditions:

(1) f is hemicontinuous on $B(0, r)$,

(2) f is monotone on $B(0, r)$,

(3) for any $x \in \partial D$ (the boundary of D), $\langle x, f(x)\rangle \geq 0$,

then there is at least one element $x_0 \in D$ such that $f(x_0) = 0$.

Proof. A proof of this result is in [23]. □

A consequence of Theorem 5.6 is the following result.

Theorem 5.7. *If $f : H \to H$ is cocoercive with modulus $\beta > 0$, $\alpha > \dfrac{1}{4\beta}$ and if there is $r_0 > 0$ such that for any $x \in H$ with $\|x\| = r_0$ we have*

$$\langle x, \mathcal{N}_\alpha(\Omega, f)(x)\rangle \geq 0.$$

Then there exists x_* with $\|x_*\| \leq r_0$ and $\mathcal{N}_\alpha(\Omega, f)(x_*) = 0$.

Let $(E, \|\cdot\|)$ be a Banach space and $r > 0$.

We denote $\overline{B}_r = \{x \in E \mid \|x\| \leq r\}$ and $S_r = \{x \in E \mid \|x\| = r\}$. We say that a mapping $G : E \times E \to \mathbb{R}$ satisfies properties (g_1) and (g_2) for the real number r if:

(g_1) $G(x,x) \geq 0$ for any $x \in S_r$,

(g_2) $G(\lambda x, y) \geq \lambda G(x,y)$ for any $\lambda > 0$ and any $x, y \in S_r$.

If $f : E \to E$ is a mapping, we say that the equation $f(x) = 0$ is almost solvable if and only if $0 \in \overline{f(\overline{B_r})}$.

Theorem 5.8. [Isac-Avramescu] [17], [18] *Let $(E, \|\cdot\|)$ be a Banach space. Suppose that:*

(i) $f : E \to E$ is completely continuous,

(ii) $G : E \times E \to \mathbb{R}$ satisfies properties (g_1) and (g_2) for a particular $r > 0$,

(iii) $G(f(x), x) < 0$ for any $x \in S_r$,

then the equation $f(x) = 0$ is almost solvable in $\overline{B}_r$. If E is finite dimensional then the equation $f(x) = 0$ is solvable.

From Theorem 5.8. we deduce the following result.

Theorem 5.9. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $f : H \to H$ a continuous mapping and $\Omega \subset H$ a non-empty closed convex subset.*

If $\varphi : H \to H$ is completely continuous mapping and there exists $r > 0$ such that

$$\langle \mathcal{N}_\varphi(\Omega, f)(x), x \rangle < 0 \textit{ for any } x \in S_r,$$

then the equation $\mathcal{N}_\varphi(\Omega, f)(x) = 0$ is almost solvable in $\overline{B_r}$.

Remark. We can show that the almost solvability of the equation $\mathcal{N}_\varphi(\Omega, f)(x) = 0$ has the following interpretation. For any $\varepsilon > 0$ there exist y_ε with $\|y_\varepsilon\| < \varepsilon$ and an element $x_\varepsilon \in \overline{B_r}$ such that the element $x_*^\varepsilon = P_\Omega(\varphi(x_\varepsilon))$ is a solution of the perturbed variational inequality $VI(f - y_\varepsilon, \Omega)$, where $f - y_\varepsilon$ is the mapping $(f - y_\varepsilon)(x) = f(x) - y_\varepsilon$.

Open subjects

(1) It is useful to study from several points of view the normal operators (directional differentiability, quasi-boundedness, pseudo-monotonicity etc.)

(2) It is interesting to find new solvability theorems for equations $\mathcal{N}_\alpha(\Omega, f)(x) = 0$ and $\mathcal{N}_\varphi(\Omega, f)(x) = 0$.

6. Comments

We presented in this paper three new classes of nonlinear operators related to the study of solvability of complementarity problems and variational inequalities.

The study from several point of view of nonlinear operators presented in this paper may be new subjects in nonlinear analysis. In this sense we put in evidence some open subjects.

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