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ON SINC METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The Sinc-Galerkin method is presented as a very useful numerical method for partial differential equations. A problem with exact solution from numerical oceanography is used to explore the accuracy and exponential convergence of expansions using composite translated sinc functions as a basis set. The absolute or convective nature of the instability, modified by numerical effects in numerical simulation of unstable flows is also studied by numerical experiments.
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1. Introduction

Many numerical methods for partial differential equations are based on exact relationships that polynomials satisfy. These procedures, like Chebyshev-collocation, Fourier-Galerkin or Legendre-tau generally do very well, with spectral accuracy, in a region where the solution to be approximated is analytic and very poorly in a neighborhood of a singularity of this solution.

Conversely, the *sinc* methods, based on the *sinc* functions and the translates as basis functions, have the same accuracy whether or not the solution to be approximated has a singularity (or almost singularity) at the boundary of the domain.

Recently, sinc function methods have been developed to the point where time-dependent fluid dynamics models could be solved by the Sinc-Galerkin approach [1-3]. This procedure may prove to be a useful complement to finite element and time stepping methods currently used in such models.

The Sinc-Galerkin method is described in some detail, following [4], in Section 2. Section 3 presents a two-dimensional Stommel ocean model to illustrate the accuracy and computational speed of the method. The effect of the spatial discretization by sinc functions in the numerical simulation of a parallel one-dimensional unstable flow, precisely changing the absolute or convective nature of the instability which can result in a wrong global dynamics of the flow, is presented in Section 4.

2. Sinc methods

Let $h > 0$ and let $B(h)$ denote the family of functions f that are analytic in the entire complex plane $\mathbb{C}$, such that

$$|f(z)| \leq Ce^{\pi|z|/h}$$

and such $f \in L^2(\mathbb{R})$.

Corresponding to a function f defined on $\mathbf{R}$, the Whittaker's (E.T.Whittaker 1915, 1927 and J.M.Whittaker 1935) cardinal function $C(f,h)$ is defined by

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)S(k,h,x) \tag{1}$$

whenever this series converges, where $h > 0$ is the stepsize and where

$$S(k,h,x) = \frac{\sin\left[(\pi/h)\,(x-kh)\right]}{(\pi/h)\,(x-kh)} \tag{2}$$

In the case of the interval $[-1,1]$, instead of the basis functions (2) we take

$$S(k,h,\Phi\,(x)) = \frac{\sin\left[(\pi/h)\,(\Phi\,(x)-kh)\right]}{(\pi/h)\,(\Phi\,(x)-kh)} \tag{3}$$

where $\Phi\,(z) = \log\left(\frac{1+z}{1-z}\right)$.

Set

$$\delta_{jk}^{(0)} = \begin{cases} 1 & if \quad j=k \\ 0 & if \quad j \neq k \end{cases} \tag{4}$$

$$\delta_{jk}^{(1)} = \begin{cases} 0 & if \quad j=k \\ \frac{(-1)^{k-j}}{k-j} & if \quad j \neq k \end{cases} \tag{5}$$

$$\delta_{jk}^{(2)} = \begin{cases} -\frac{\pi^2}{3} & if \quad j=k \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & if \quad j \neq k \end{cases} \tag{6}$$

Theorem 1. *([4]) Let $f \in B(h)$. Then*

$$f(z) = C(f,h)(z) \quad for \quad all \quad z \in \mathbb{C} \tag{7}$$

$$f' \in B(h) \tag{8}$$

$$f^{(n)}(kh) = h^{-n} \sum_{j=-\infty}^{\infty} \delta_{jk}^{(n)} f(jh) \tag{9}$$

For the approximations over the real line, let $d > 0$ and

$$D_d = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < d\} \tag{10}$$

Let $p \geq 1$ and let $B_p(D_d)$ denote the family of functions f that are analytic in D_d, such that

$$\int_{-d}^{d} |f(x+iy)|\, dy \to 0 \quad as \quad x \to \pm\infty$$

and such that

$$N_p(f, D_d) \equiv \lim_{\gamma \to d^-} \left\{ \left(\int_R |f(x+iy)|^p\, dx \right)^{1/p} + \left(\int_R |f(x-iy)|^p\, dx \right)^{1/p} \right\} < \infty$$

Theorem 2. *([4]) Let $n \geq 0$, $f \in B_p(\mathrm{D}_d)$ with $p = 1$ or $p = 2$, $|f(x)| < Ce^{-\alpha|x|}$ for all $x \in \mathbb{R}$, where C and α are positive constants. Let $\pi d/h > 1$. Then by choosing $h = [\pi d/(\alpha N)]^{1/2}$ we have*

$$\left| f^{(n)}(kh) - h^{-n} \sum_{j=-N}^{N} \delta_{jk}^{(n)} f(jh) \right| \leq C_1 N^{(n+1)/2} e^{-(\pi d \alpha N)^{1/2}} \tag{11}$$

For the approximations over finite intervals, if F is analytic and bounded in the domain

$$\mathcal{D} = \left\{ z : \left| \arg\left(\frac{1+z}{1-z} \right) \right| < d \right\} \tag{12}$$

and

$$|F(x)| \leq C(1+x)^{\alpha}(1-x)^{\alpha} \qquad \text{on } [-1,1]$$

where $\alpha > 0, C > 0$, then if we take $h = [\pi d/(\alpha N)]^{1/2}$ and $z_k = \frac{e^{kh}-1}{1+e^{kh}}$ we have the evaluation of the interpolating error

$$\left| F(x) - \frac{h}{\pi} \sin\left\{ \frac{\pi}{h} \Phi(x) \right\} \sum_{k=-N}^{N} \frac{(-1)^k F(z_k)}{\log\left((1+x)/(1-x)\right) - kh} \right| \leq C_1 N^{1/2} e^{-(\pi d \alpha N)^{1/2}}$$

where $\Phi(x) = \log\left(\frac{x}{1-x}\right)$. If F does not vanish at 0 or at 1, we take the function

$$G(x) = F(x) - (1-x)F(0) - xF(1) \tag{13}$$

instead of F.

In the case of Dirichlet problems, the sinc approximation procedures are particularly powerful. As an example, let us consider the Dirichlet problem

$$(Lf)(x) \equiv f''(x) + \mu(x) f'(x) + \upsilon(x) f(x) - \sigma(x) = 0, \qquad x \in \Gamma = (-1,1) \tag{14}$$
$$f(-1) = f(1) = 0$$

Let us assume that μ, ν, σ are analytic in the region $\mathcal{D}$, that the Dirichlet problem (14) has an unique solution f which is analytic in $\mathcal{D}$, such that

$$\int_{\partial\mathcal{D}} \left| \frac{f''(z)}{\Phi'(z)} dz \right| < \infty, \qquad \int_{\partial\mathcal{D}} \left| \frac{\mu(z) f'(z)}{\Phi'(z)} dz \right| < \infty$$
$$\int_{\partial\mathcal{D}} \left| \frac{\upsilon(z) f(z)}{\Phi'(z)} dz \right| < \infty, \quad \int_{\partial\mathcal{D}} \left| \frac{\sigma(z)}{\Phi'(z)} dz \right| < \infty$$

and such that

$$|f(x)| \leq Ce^{-\alpha|\Phi(x)|}$$

on $[-1, 1]$.

We approximate f on $(-1, 1)$ by

$$f(x) \cong f_N(x) = \sum_{k=-N}^{N} f_k S(k, h) \circ \Phi(x) \tag{15}$$

The Galerkin scheme enables us to determine the coefficients $f_k \cong f(z_k)$ by solving the linear system of equations

$$\int_{-1}^{1} (Lf_N)(x)\, S(k, h) \circ \Phi(x) \frac{1}{\Phi'(x)} dx = 0, \qquad k = -N, ..., N$$

By integration by parts to change integrals involving derivatives of f_N into integrals involving f_N, the above explicit system can be obtained if simply we make the following replacements into the differential equation (14):

$$\frac{\upsilon(z_k) f(z_k)}{\Phi'(z_k)^2} = \frac{\upsilon(z_k) f_k}{\Phi'(z_k)^2} + E_1$$

$$\frac{\mu(z_k) f'(z_k)}{\Phi'(z_k)^2} = -\left\{ \frac{(\mu/\Phi')'(z_k)}{\Phi'(z_k)} f_k + \frac{1}{h} \sum_{j=-N}^{N} \delta_{kj}^{(1)} \left(\frac{\mu(z_j)}{\Phi'(z_j)} \right) f_j \right\} + E_2$$

$$\frac{f''(z_k)}{\Phi'(z_k)^2} = \left\{ \frac{(1/\Phi')''(z_k)}{\Phi'(z_k)} f_k + \frac{1}{h^2} \sum_{j=-N}^{N} \left[\delta_{kj}^{(2)} - \frac{h\delta_{kj}^{(1)} \Phi''(z_j)}{\Phi'(z_j)^2} \right] f_j \right\} + E_3$$

where

$$E_1 = O(N^{-1/2} e^{-(\pi d \alpha N)^{1/2}})$$

$$E_2 = O(e^{-(\pi d \alpha N)^{1/2}})$$

$$E_3 = O(N^{1/2} e^{-(\pi d \alpha N)^{1/2}})$$

3. An ocean model [5]

Stommel designed an ocean model to explain the westward intensification of wind-driven ocean currents. Consider a rectangular ocean with the x and y axes point eastward and northward respectively. The boundaries of the ocean are at $x = 0, \lambda$ and $y = 0, b$. The ocean is considered as a homogeneous and incompressible layer of constant depth D. A streamfunction ψ is defined by

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

where u and v are the x and y components of the velocity vector.

The surface wind stress is taken as $-F\cos(\pi y/b)$. The component frictional forces are taken as $-Ru$ and $-Rv$ where R is the frictional coefficient. The Coriolis parameter $f = f(y)$ and his latitudinal variation $\beta = \frac{df}{dy}$ are also introduced. Under these conditions Stommel derived an equation for the streamfunction ψ

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \alpha\frac{\partial\psi}{\partial x} = -\gamma\sin\left(\frac{\pi}{b}y\right) \tag{16}$$

with the boundary conditions

$$\psi(0,y) = \psi(\lambda,y) = \psi(x,0) = \psi(x,b) = 0$$

The two parameters α and γ are defined by

$$\alpha = \frac{D\beta}{R}, \quad \gamma = \frac{F\pi}{Rb}$$

The physical parameters are

$$\begin{aligned} &\lambda = 10^7 m, \quad b = 2\pi\times 10^6 m, \quad D = 200m \\ &F = 0.3\times 10^{-7} m^2 s^{-2}, R = 0.6\times 10^{-3} m s^{-1}, \beta = 10^{-11} m^{-1} s^{-1} \end{aligned} \tag{17}$$

The grid spacings $(\Delta x, \Delta y)$ in most ocean numerical models are not small. For example, a global ocean model is considered having a high resolution when a horizontal grid is approximately 14.5 km. For such large grid spacing, use of highly accurate schemes becomes urgent.

Chu and Fan ([5]) used a three-point combined compact difference scheme to reduce the errors in the ocean model. To explore the accuracy and exponential convergence of expansions using composite translated sinc functions as a basis set, in the present paper we compare the sinc-numerical solution with the exact solution of the problem

$$\psi = -\gamma\left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi}{b}y\right)\left(pe^{Ax} + qe^{Bx} - 1\right)$$

where

$$A = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \left(\frac{\pi}{b}\right)^2}, \quad B = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + \left(\frac{\pi}{b}\right)^2}$$

$$p = \frac{1 - e^{B\lambda}}{e^{A\lambda} - e^{B\lambda}}, \quad q = 1 - p$$

We transform the domain $[0,\lambda]\times[0,b]$ into $[-1,1]\times[-1,1]$ by a linear change of variables and we look for the numerical solution in the form

$$u(x,y) = \sum_{i=-N}^{N}\sum_{j=-N}^{N} u_{i,j} S(i,h)\circ\Phi(x) S(j,h)\circ\Phi(y)$$

When the Galerkin method is applied to the Stommel equation (16), we have for the first term to compute

$$\sum_{i,j=-N}^{N} \int_{-1}^{1} \frac{[S(i,h)\circ\Phi(x)]'' \, S(k,h)\circ\Phi(x)}{\Phi'(x)} dx \cdot$$
$$\int_{-1}^{1} \frac{S(j,h)\circ\Phi(y) S(l,h)\circ\Phi(y)}{\Phi'(y)} dy$$

Following the previous section, the result is

$$\frac{h^2}{\Phi'^2(y_l)} (B\cdot u)_{kl}$$

thus u_{xx} must be replaced by $h^2 B\cdot u\cdot A$ to obtain the linear system for u_{ij}. Here,

$$B = diag\left(\frac{\left(\frac{1}{\Phi'}\right)''(x_k)}{\Phi'(x_k)}\right) + \frac{1}{h}\delta^{(1)} diag\left(\left(\frac{1}{\Phi'}\right)'(x_k)\right) + \frac{1}{h^2}\delta^{(2)}$$

$$A = diag\left(\frac{1}{\Phi'^2(y_l)}\right), \quad u = (u_{kl})_{k,l=-N\ldots N}$$

Here

$$x_k = \frac{e^{kh}-1}{1+e^{kh}}, \quad y_l = \frac{e^{lh}-1}{1+e^{lh}}$$

for $k, l = -N, ..., N$.

Similarly, u_{yy} must be replaced by $h^2 A\cdot U\cdot B^T$, where ^{T} denotes the transpose of the matrix. Finally, u_x must be replaced by $h^2 C\cdot u\cdot A$, where

$$C = -\left[diag\left(\frac{\left(\frac{1}{\Phi'}\right)'(x_k)}{\Phi'(x_k)}\right) + \frac{1}{h}\delta^{(1)} diag\left(\frac{1}{\Phi'(x_k)}\right)\right]$$

and the right hand side F by $h^2 A\cdot F\cdot A$.

The matrix form of this linear system for the unknowns u_{ij} is

$$A^{-1}Bu + u\left(A^{-1}B\right)^T + \alpha A^{-1}Cu = F$$

If we take $N = 12$ (this means 25 points in the linear grid), $h = 0.8$ we obtain a numerical solution with an average relative error of order 1.6×10^{-3}, see figures 1 and 2. The same order of relative error was obtained in [5] for $N = 50$ but the computing time is much shorter for sinc method than for the compact difference scheme.

The MATLAB programs are

```
tic;N=12;h=0.8;[x,A,B,C]=matr(N,h);
% problema oceanului Stommel cu metoda sinc
M=A\B;P=A\C;k=-N:N;w=h*k;xk=(exp(w)-1)./(exp(w)+1);
[X,Y]=meshgrid(xk,xk);
F=-sin(pi/2*(Y'+1));
I=eye(2*N+1);
U=((16.e-4)*kron(I,M)+(4.e-2)/pi^2*kron(M,I)+...
(8.e-2)/3*kron(I,P))\reshape(F,(2*N+1)^2,1);
u=reshape(U,2*N+1,2*N+1);toc;
```

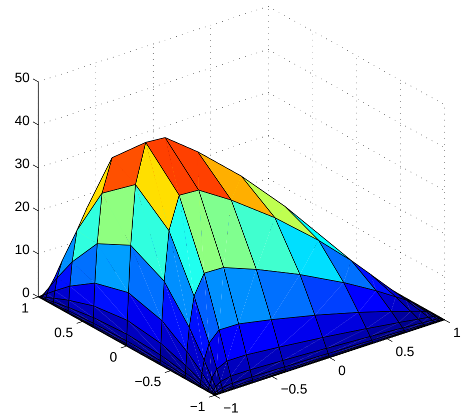

FIGURE 1. Streamfunction from Stommel ocean model with sinc method

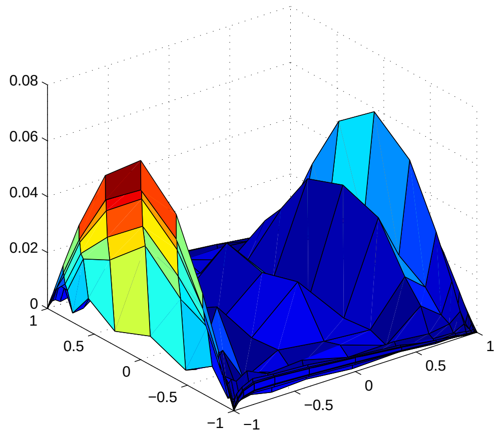

FIGURE 2. Errors for the sinc method

```
surf(X′,Y′,u);pause;
cb=2*pi*10^6;cd=200;cf=0.3/10^7;cr=0.6/10^3;
ca=cd/10^11/cr;cg=cf*pi/cr/cb;cl=10^7;
A=-ca/2+sqrt(ca^2/4+(pi/cb)^2);B=-ca/2-sqrt(ca^2/4+(pi/cb)^2);
p=(1-exp(B*cl))/(exp(A*cl)-exp(B*cl));q=1-p;
uex=-cg*(cb/pi)^2*sin(pi/2*(Y'+1)).*(p*exp(A*(cl/2*(X'+1)))+...
q*exp(B*(cl/2*(X′+1)))-1);
surf(X′,Y′,abs(u-uex));
```
and
```
function [x,A,B,C]=matr(N,h)
```

```
k=-N:N;w=k*h;x=(exp(w)-1)./(exp(w)+1);
A=diag(4*exp(2*w)./((1+exp(w)).^4));
I1=zeros(2*N+1);I2=-pi^2/3*ones(2*N+1);
for j=1:2*N+1
 for k=1:2*N+1
 if j~=k I1(j,k)=(-1)^(k-j)/(k-j);I2(j,k)=-2*(-1)^(k-j)/(k-j)^2;
end
 end
end
B=diag(-2*exp(w)./((1+exp(w)).^2))+...
I1*diag((1-exp(w))./(1+exp(w)))/h+I2/h^2;
C=-diag(2*exp(w).*(1-exp(w))./((1+exp(w)).^3))-...
I1*diag(2*exp(w)./((1+exp(w)).^2))/h;
```

4. Numerical effects of the sinc method

In the last years the notions of local/global and absolute/convective instabilities have been recognized as essential for understanding the spatio-temporal dynamics of open flows. In the laboratory frame, a convectively unstable flow will relax everywhere to the basic state as the transient is advected downstream. By contrast, in an absolute unstable flow, a transient will initially grow in place and then saturate, leading to self-sustained oscillations.

The numerical simulation of unsteady flows has become a routine task in many fields of science and technology. In a numerical simulation, changing the absolute or convective nature of the instability, responsible for the time-dependent behavior, can result in the wrong global dynamics of the flow. it is important to ensure that, at least locally, the nature of the instability is not changed by numerical effects.

In this section we analyze the simplest case of a numerical simulation of a parallel one-dimensional unstable flow and we illustrate this on the linearized Ginzburg-Landau operator, which has been considered to model the transition of closed or open fluid dynamical systems,

$$u_t = \mu u - U u_x + \gamma u_{xx} \tag{18}$$

Here U is the mean (positive) advection velocity, γ is the (positive) diffusion coefficient and μ is the bifurcation parameter. If we consider solutions $u(x,t)$ in the form of normal modes $Ae^{i(kx-\omega t)}$, where k and ω are the complex spatial wavenumber and temporal frequency respectively, we obtain the physical dispersion relation

$$\omega = Uk + i(\mu - \gamma k^2)$$

In the dimensionless variables $\widetilde{\omega} = \omega\gamma/U^2$, $\widetilde{k} = k\gamma/U$ and parameters $\widetilde{\mu} = \mu\gamma/U^2$, $R = \Delta x U/\gamma$ and $S = \Delta t U^2/\gamma$, the physical dispersion relation becomes

$$\widetilde{\omega} = \widetilde{k} + i(\widetilde{\mu} - \widetilde{k}^2)$$

A stable flow admits only damped modes, $\mathrm{Im}\left(\widetilde{\omega}(\widetilde{k})\right) < 0$ for every real wavenumber $\widetilde{k}$, otherwise it is unstable. In the dispersion relation we have a bifurcation from a stable to an unstable behavior for $\widetilde{\mu} = 0$. For $0 < \widetilde{\mu} < 1/4$ the flow will be physically convectively unstable and for $1/4 < \widetilde{\mu} < 1/2$ it will be physically absolutely unstable.

In the paper [6], Cossu and Loiseleux performed a stability analysis of the numerical finite difference schemes Euler Explicit, Crank-Nicholson and Euler Implicit and found the regions for the parameters R and S for convective or absolute instability. It is remarkable how easily results affected from numerical transition could be confused with physically correct results.

In this section, we perform a similar analysis if the spatial part of the differential operator (18) is discretized by the sinc method. We consider this equation for $x \in (-\infty, \infty)$ with homogeneous Dirichlet boundary conditions $u(\pm\infty, t) = 0$. In order to detect the absolute or convective nature of the numerical solution we analyze the evolution of the discretized Green function G, i.e. the evolution of an initial condition having value 1 for $x = 0$ and zero everywhere else. The instability is convective if, for sufficiently large n, $|G(0, t_n)|$ decreases and it is absolute if it increases. The numerical results are obtained with $U = 1, \gamma = 1$ and for $\mu < 0.25$ (physically convective unstable) and $\mu > 0.25$ (physically absolute unstable) flow for the Euler Implicit method.

With the notations from the previous sections, the discrete form of the problem is

$$\frac{u^{n+1} - u^n}{\Delta t} = \left[\mu\delta^{(0)} - \frac{1}{h}\delta^{(1)} + \frac{1}{h^2}\delta^{(2)}\right] u^{n+1}$$

where $u^n = (u(-Nh, t_n), ..., u(Nh, t_n))^T$. In the matrix form, the problem is

$$(I - M\Delta t)u^{n+1} = u^n$$
$$u^0 = \frac{\sin(\frac{\pi}{h}x)}{\frac{\pi}{h}x}$$

where $x = (-Nh, ..., Nh)^T$.

For $N = 16$, $h = 0.6$ for all $\Delta t \in (0, 4)$ for example, if $\mu \in (0.2, 0.25)$ the numerical flow is convectively unstable and if $\mu \in (0.26, 0.4)$ the flow is absolutely unstable. Such results may change with h (see figure 3) but they are much better than the results for the finite difference discretization ([6]). Consequently, one can better control the numerical effects of the sinc method on the global dynamics of the flow. The optimal choice of h is stil an open problem.

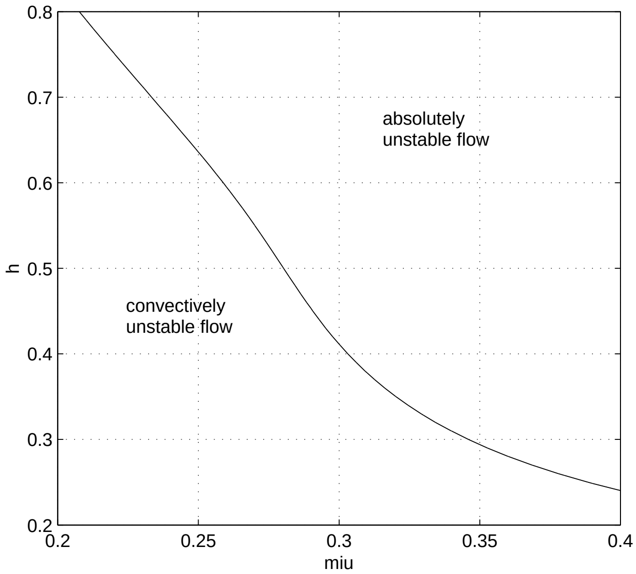

FIGURE 3. The convectively and absolutely unstable regions

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