Seminar on Fixed Point Theory Cluj-Napoca, Volume 3, 2002, 381-388
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

THE GOURSAT-IONESCU PROBLEM FOR HYPERBOLIC INCLUSIONS WITH MODIFIED ARGUMENT

GEORGETA TEODORU

Department of Mathematics,
Technical University "Gh. Asachi" Iaşi,
11 Carol I Blvd, Ro-6600, Iaşi 6, ROMANIA
E-mail: teodoru@math.tuiasi.ro

Abstract. In this paper we consider the Goursat- Ionescu Problem defined in Straburzyński's sense, for hyperbolic inclusions with modified argument. An existence theorem for a local solution of this problem is proved and some properties of the set of its solutions are established.
Keywords: multifunction, hyperbolic inclusion, upper-semicontinuity, initial values, absolutely continuous in Carathéodory's sense function.
AMS Subject Classification: 35L15, 35R70.

1. Introduction

Goursat's Problem defined by E. Goursat [11] for a quasilinear hyperbolic equation consists in determining one of its solutions, provided that the values of the solution on two curve arcs having a common point, which may be taken as the origin of the system of coordinates, are known [4].

In his PhD Thesis (1927) [13], D.V. Ionescu studied − for the first time in the mathematical literature − boundary value problems of Darboux, Cauchy, Picard and Goursat types for second order partial differential equations with modified argument.

More recently, a series of authors studied the same problems for second order hyperbolic equations of various forms.

In this paper, we consider Goursat-Ionescu Problem in Straburzyński's sense [18], for a hyperbolic inclusion with modified argument.

Let a, b, a', b', a_0, b_0 be positive numbers with $0 < a_0 \leq a'$, $0 < b_0 \leq b'$ and $y = g(x) : [0, a] \to I\!R$, $x = h(y) : [0, b] \to I\!R$ be nondecreasing functions of class C^1 such that $g(0) = h(0) = 0$, $0 \leq g(x) \leq b$, $0 \leq h(y) \leq a$. We denote:

$$P = [-a', a] \times [-b', b], \quad \Delta = [0, a] \times [0, b], \quad \Delta_0 = [0, x_0] \times [0, y_0] \subseteq \Delta,$$

$$D = \{(s, t)/h(t) < s \leq a, \quad g(s) < t \leq b\}, \quad P_0 = [-a_0, x_0] \times [-b_0, y_0] \subseteq P,$$

$$D_{xy} = \{(s, t)/h(t) < s \leq x, \quad g(s) < t \leq y\}$$

for $(x, y) \in \Delta$, $G = P - D$, $G_0 = P_0 - D_{x_0 y_0}$, $G_0 \subseteq G$.

Let $\varphi : P \to I\!R^n$ be an absolutely continuous function in Carathéodory's sense, $\varphi \in C^*(P; I\!R^n)$ [1,§565 - §570].

We consider Goursat-Ionescu Problem for the hyperbolic inclusion with modified argument of the form

$$\frac{\partial^2 z(x,y)}{\partial x \partial y} \in F(x,y,z(\alpha(x,y),\beta(x,y))), \ \ (x,y) \in \overline{D}, \tag{1.1}$$

$$z(x,y) = \varphi(x,y), \ \ (x,y) \in G, \tag{1.2}$$

where $F : \Delta \times \Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex and non-empty values, $\Omega \subset \mathbb{R}^n$ is an open subset, $\alpha \in C(\Delta; [0,a]), \ \ \beta \in C(\Delta; [0,b])$.

Under suitable assumptions, we prove an existence theorem for a local solution of this problem, and that the set of solutions is compact in Banach space $C(P_0; \mathbb{R}^n)$, $P_0 = [-a_0, x_0] \times [-b_0, y_0] \subseteq P$; moreover, as a function of the initial values, this set defines an upper- semicontinuous multifunction.

This study was suggested by papers which deal with the Goursat Problem [7], [18], with Goursat-Ionescu Problem for univalued hyperbolic equations [8], [9] and [19].

2. Preliminaries

The definitions and Theorem 2.1 in this section are taken from [1], [2], [3], [5]-[7], [14]-[17].

Definition 2.1. Let X and Y be two non-empty sets. A *multifunction* $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup\limits_{x \in X} \Phi(x)$ is the *range* of Φ.

Definition 2.2. Let us consider $\Phi : X \to 2^Y$.
a) If $A \subset X$, the *image* of A by Φ is $\Phi(A) = \bigcup\limits_{x \in A} \Phi(x)$;
b) If $B \subset Y$, the *counterimage* of B by Φ is $\Phi^-(B) = \{x \in X | \Phi(x) \cap B \neq \emptyset\}$;
c) The *graph* of Φ, denoted *graph* Φ is the set

$$graph\Phi = \{(x,y) \in X \times Y | y \in \Phi(x)\}.$$

Definition 2.3. Let now take $\Phi : X \to 2^X$. An element $x \in X$ with the property $x \in \Phi(x)$ is called a *fixed point* of the multifunction Φ.

Definition 2.4. A univalued function $\varphi : X \to Y$ is said to be a *selection* of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

Definition 2.5. Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is *upper-semicontinuous* if, for any closed subset $B \subset Y$, $\Phi^-(B)$ is closed in X.

Definition 2.6. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is *measurable* if $\Phi^-(B) \in \mathcal{F}$ for every closed subset $B \subset Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^-(B)$ is measurable.

Theorem 2.1. [17]. Let X and Y be two compact metric spaces and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$.

The following assertions are equivalent:
(i) the multifunction Φ is upper-semicontinuous;
(ii) the graph of Φ is a closed subset of $X \times Y$;
(iii) any would be the sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, from $x_n \to x$, $y_n \in \Phi(x_n)$, $y_n \to y$ it follows $y \in \Phi(x)$.

Definition 2.7. [5]-[6]. The function $u : \Delta \to \mathbb{R}^n$ is *absolutely continuous in Carathéodory's sense* [1, §565 - §570] iff $u(x,y)$ is continuous on Δ, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x(x,y)$ is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesgue-integrable on Δ.

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(\Delta;\mathbb{R}^n)$ [5]-[6].

3. Results

In a similar way as in [2] and [19], we define the notion of a *local solution* for the Goursat-Ionescu Problem (1.1)+(1.2) and we prove an existence theorem for a local solution of this problem, together with some properties of the set of solutions, namely that this is a compact subset in Banach space $C(\Delta_0;\mathbb{R}^n)$ and, as a function of initial values, it defines an upper-semicontinuous multifunction.

Let the following hypotheses be satisfied:
(H_0) The curves $C_1 : y = g(x),\quad 0 \le x \le a$, and $C_2 : x = h(y),\quad 0 \le y \le b$ are defined by nondecreasing functions of class C^1 such that $g(0) = h(0)$, $0 \le g(x) \le b$, $0 \le h(y) \le a$.
(H_1) $F : \Delta \times \Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex, non-empty values in $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is an open subset, $\Delta = [0,a] \times [0,b] \subset \mathbb{R}^2$.
(H_2) For any $(x,y) \in \Delta$, the mapping $z \to F(x,y,z)$ is upper-semicontinuous on Ω;
(H_3) For any $z \in \Omega$ the mapping $(x,y) \to F(x,y,z)$ is Lebesgue measurable on Δ;
(H_4) $\alpha \in C(\Delta;[0,a])$ and $\beta \in C(\Delta;[0,b])$;
(H_5) There exists a function $k : \Delta \to \mathbb{R}_+$, $k \in \mathcal{L}^1(\Delta;\mathbb{R}_+)$ such that

$$\|\zeta\| \le k(x,y) \ \text{ for } \ \forall \zeta \in F(x,y,z),\ \ \forall (x,y) \in \Delta,\ \ \forall z \in \Omega;$$

(H_6) There exists a convex, compact set $M \subset \Omega$ and a point $(x_0,y_0) \in]0,a] \times]0,b]$, such that

$$\int_0^{x_0}\int_0^{y_0} k(s,t)ds\, dt \le d(M, C_\Omega),$$

where $d(M,C_\Omega)$ is the distance from M to $C_\Omega = \mathbb{R}^n - \Omega$;
(H_7) The function $\varphi : P \to \mathbb{R}^n$ is absolutely continuous in Carathéodory's sense, $\varphi \in C^*(P;\mathbb{R}^n)$.
(H_8) The values of function $\lambda : \Delta \to \mathbb{R}^n$, defined by

$$\lambda(x,y) = \varphi(0,0) + \int_0^x \varphi'_x(s,g(s))ds + \int_0^y \varphi'_y(h(t),t)dt, \tag{3.1}$$

belong to the set M for $(x,y) \in \Delta_0 = [0,x_0] \times [0,y_0] \subset \Delta$.

Remark. It follows that the function λ defined by (3.1) is absolutely continuous in Carathéodory's sense [1, §565 - §570], $\lambda \in C^*(\Delta; I\!R^n)$, due to hypotheses (H_7), $\varphi \in C^*(P; I\!R^n)$ and the integral is absolutely continuous.

Definition 3.1. The *Goursat-Ionescu Problem* for the hyperbolic inclusion with modified argument (1.1) means to determine a *solution* of this inclusion which satisfies the initial conditions (1.2).

Definition 3.2. It is defined a *local solution* of the Goursat-Ionescu Problem (1.1)+(1.2) as a function $Z : P_0 \to \Omega$, $P_0 = [-a_0, x_0] \times [-b_0, y_0]$, with $0 < a_0 \leq a'$ and $0 < b_0 \leq b'$, which is absolutely continuous in Carathéodory's sense [1], $Z \in C^*(D_0; I\!R^n)$ and satisfies (1.1) a.e. for $(x, y) \in \overline{D}_{x_0,y_0}$ and also conditions (1.2) for $(x, y) \in G_0 = P_0 - D_{x_0y_0} \subseteq G$.

Theorem 3.1. Let the hypotheses $(H_0) - (H_8)$ be satisfied. Then:
(i) there exists at least a local solution Z of the Goursat-Ionescu Problem (1.1)+(1.2);
(ii) the set S_λ of local solutions Z is compact in the Banach space $C(P_0; I\!R^n)$;
(iii) the multifunction $\lambda \to S_\lambda$ is upper-semicontinuous on $C^*(\Delta_0; I\!R^n)$ taking values in $C(\Delta_0; I\!R^n)$.

Proof. (i) Let $C^*(P_0; I\!R^n)$ be the set of absolutely continuous functions in Carathéodory's sense defined on P_0 with values in $I\!R^n$ [1]. We denote by $\mathcal{Z}_M$ the set of functions $Z : P_0 \to I\!R^n$, $Z \in C^*(P_0; I\!R^n)$, which satisfy the inequality

$$\left\|\frac{\partial^2 Z(x,y)}{\partial x \partial y}\right\| \leq k(x,y), \quad \text{a.e. for } (x,y) \in \overline{D}_{x_0y_0}, \tag{3.2}$$

and also conditions (1.2) for $(x, y) \in G_0 = P_0 - D_{x_0y_0}$. The notation $\mathcal{Z}_M$ is suitable because, by hypothesis (H_8), $\lambda(x, y) \in M$ for $(x, y) \in \Delta_0$. We remark that the absolute continuity in Carathéodory's sense of Z assures the existence of the derivative $\dfrac{\partial^2 Z(x,y)}{\partial x \partial y}$ a.e. for $(x, y) \in P_0$ [1, §565 - §570].

We have $\mathcal{Z}_M \subset C^*(P_0; I\!R^n)$. Then, by hypothesis (H_6) and inequality (3.2), for any $Z \in \mathcal{Z}_M$, it follows that $Z(x, y) \in \Omega$.

Indeed, integrating $\dfrac{\partial^2 Z(x,y)}{\partial x \partial y}$ on $\overline{D}_{xy}$ we obtain

$$\begin{aligned} Z(x,y) &= \varphi(0,0) + \int_0^x \varphi'_x(s, g(s))ds + \int_0^y \varphi'_y(h(t), t)dt + \iint_{\overline{D}_{xy}} \frac{\partial^2 Z(s,t)}{\partial s \partial t} ds\, dt = \\ &= \lambda(x,y) + \iint_{\overline{D}_{xy}} \frac{\partial^2 Z(s,t)}{\partial s \partial t} ds\, dt. \end{aligned} \tag{3.3}$$

Using hypothesis (H_6), inequality (3.2) and (3.3) it results

$$\begin{aligned} \|Z(x,y) - \lambda(x,y)\| &= \left\|\iint_{\overline{D}_{xy}} \frac{\partial^2 Z(s,t)}{\partial s \partial t} ds\, dt\right\| \leq \iint_{\overline{D}_{xy}} \left\|\frac{\partial^2 Z(s,t)}{\partial s \partial t}\right\| ds\, dt \leq \\ &\leq \|\iint_{\overline{D}_{xy}} k(s,t) ds\, dt \leq \int_0^{x_0}\int_0^{y_0} k(s,t) ds\, dt \leq d(M, C_\Omega). \end{aligned} \tag{3.4}$$

From the hypothesis (H_8), $\lambda(x,y) \in M$ for $(x,y) \in \Delta_0 = [0,x_0] \times [0,y_0]$ and we have

$$d(Z(x,y),\lambda(x,y)) = \|Z(x,y) - \lambda(x,y)\| \leq d(M, C_\Omega), \tag{3.5}$$

which shows that $Z(x,y) \in \Omega$ for $(x,y) \in \Delta_0$.

The set of functions $\mathcal{Z}_M$ is *convex* and *compact* in $C(P_0; \mathbb{R}^n)$. The convexity follows by the definition of this set, and its compactness from the Arzelà-Ascoli Theorem, using hypotheses (H_0), (H_6), (H_7), (H_8).

We denote by $\mathcal{G}$ the set of the triples $(\lambda, Z, U) \in C^*(\Delta_0; \mathbb{R}^n) \times \mathcal{Z}_M \times \mathcal{Z}_M$ with the property that Z and U satisfy the membership relation

$$\frac{\partial^2 U(x,y)}{\partial x \partial y} \in F(x,y,Z(\alpha(x,y),\beta(x,y))), \quad \text{a.e. for } (x,y) \in \overline{D}_{x_0 y_0}. \tag{3.6}$$

We prove that, for each $\lambda \in C^*(\Delta_0; \mathbb{R}^n)$ with $\lambda(x,y) \in M$ for $(x,y) \in \Delta_0$, the set of those pairs (Z,U) such that $(\lambda, Z, U) \in \mathcal{G}$ is non-empty and the set $\mathcal{G}$ is closed.

Indeed, let us take $Z \in \mathcal{Z}_M$. From Theorem 1 [2], there exists a μ-measurable (under the μ-Lebesgue measure) multifunction $\Gamma : \Delta_0 \to 2^{\mathbb{R}^n}$ with compact, non-empty values in $\mathbb{R}^n$ such that

$$\Gamma(x,y) \subset F(x,y,Z(\alpha(x,y),\beta(x,y))), \quad \forall (x,y) \in \Delta_0. \tag{3.7}$$

Then, by Theorem 2 or Theorem 3 [3], there exists a measurable selection γ of Γ, i.e. a measurable univalued function $\gamma : \Delta_0 \to \mathbb{R}^n$ with $\gamma(x,y) \in \Gamma(x,y)$ for $(x,y) \in \Delta_0$.

Let the function $U : P_0 \to \mathbb{R}^n$ be defined by

$$U(x,y) = \begin{cases} \lambda(x,y) - \displaystyle\iint_{\overline{D}_{xy}} \gamma(s,t)ds\, dt, & (x,y) \in \overline{D}_{x_0 y_0}, \\ \varphi(x,y) & ,(x,y) \in G_0 = P_0 - D_{x_0 y_0}. \end{cases} \tag{3.8}$$

Then, the set of those pairs (Z,U) such that (λ, Z, U) is *non-empty* because

$$\gamma(x,y) \in \Gamma(x,y) \subset F(x,y,Z(\alpha(x,y),\beta(x,y))), \quad \text{a.e. for } (x,y) \in \Delta_0, \tag{3.9}$$

$$\frac{\partial^2 U(x,y)}{\partial x \partial y} = \gamma(x,y) \in \Gamma(x,y) \subset F(x,y,Z(\alpha(x,y),\beta(x,y))), \text{ a.e. for } (x,y) \in \overline{D}_{x_0 y_0}, \tag{3.10}$$

$$\left\|\frac{\partial^2 U(x,y)}{\partial x \partial y}\right\| = \|\gamma(x,y)\| \leq k(x,y), \quad \forall (x,y) \in \overline{D}_{x_0 y_0}, \tag{3.11}$$

by hypothesis (H_5) for $\zeta = \gamma(x,y)$.

For the proof that $\mathcal{G}$ is closed, we consider a sequence of elements in $\mathcal{G}$, $\{(\lambda_n, Z_n, U_n)\}_{n \in \mathbb{N}}$, convergent to (λ, Z, U) in the space $C^*(\Delta_0; \mathbb{R}^n) \times C(P_0; \mathbb{R}^n) \times L^1(P_0; \mathbb{R}^n)$. We must check that $(\lambda, Z, U) \in \mathcal{G}$, what implies, by the definition of set $\mathcal{G}$, that conditions (1.2) and (3.10) are satisfied by Z and U.

The set $\left\{\dfrac{\partial^2 U_n(x,y)}{\partial x \partial y}\right\}_{n\in I\!N}$ is *relatively weakly compact* in $L^1(\Delta_0; I\!R^n)$ by the Dunford-Pettis Criterion [10]. It follows that $\left\{\dfrac{\partial^2 U_n(x,y)}{\partial x \partial y}\right\}_{n\in I\!N}$ is weakly convergent to a function $V \in L^1(\Delta_0; I\!R^n)$. For each $(x,y) \in P_0$, we have

$$U(x,y) = \tag{3.12}$$

$$\begin{cases} w-\lim\limits_{n\to\infty} U_n(x,y) = w-\lim\limits_{n\to\infty}\left[\lambda_n(x,y) + \displaystyle\iint_{\overline{D}_{xy}} \dfrac{\partial^2 U_n(s,t)}{\partial s \partial t}\, ds\, dt\right], & (x,y)\in \overline{D}_{x_0y_0} \\ \varphi(x,y), \quad (x,y) \in G_0 = P_0 - D_{x_0y_0}. \end{cases} =$$

$$= \begin{cases} \lambda(x,y) + \displaystyle\iint_{\overline{D}_{xy}} V(s,t)\, ds\, dt, & (x,y) \in \overline{D}_{x_0y_0}, \\ \varphi(x,y), & (x,y) \in G_0. \end{cases}$$

From the weak convergence $\dfrac{\partial^2 U_n(x,y)}{\partial x \partial y} \rightharpoonup V(x,y)$, $(x,y) \in \overline{D}_{x_0y_0}$, using the Corollary of Mazur's Theorem [12], it follows that there exists a sequence of convex combinations $\{W_r\}_{r\in I\!N}$ of the set $\left\{\dfrac{\partial^2 U_r}{\partial x \partial y}, \dfrac{\partial^2 U_{r+1}}{\partial x \partial y}, ...\right\}$, strongly convergent to V in $L^1(\Delta_0; I\!R^n)$. Then, we can extract a subsequence from the sequence $\{W_r\}_{r\in I\!N}$ which converges a.e. to $V : W_{r_i} \to V$ a.e. for $(x,y) \in \Delta_0$.

Since $F(x,y,Z)$ is convex and compact for all $(x,y) \in \Delta$ and for all $Z \in \Omega$, we obtain from the previous results and from Lemma 2 [2] that

$$\begin{aligned} V(x,y) &\in \cap_{r=1}^{\infty} conv \overline{\left(\cup_{n=r}^{\infty} \dfrac{\partial^2 U_n(x,y)}{\partial x \partial y}\right)} \subset \\ &\subset \cap_{r=1}^{\infty} conv\, \overline{\left(\cup_{n=r}^{\infty} F(x,y,Z_n(\alpha(x,y),\beta(x,y)))\right)} \subset \\ &\subset F(x,y,Z(\alpha(x,y),\beta(x,y))), \quad \text{a.e. for } (x,y) \in \overline{D}_{x_0y_0}, \end{aligned} \tag{3.13}$$

from which it follows that $\mathcal{G}$ *is closed*.

Indeed, (3.13) shows that $V(x,y) \in F(x,y,Z(\alpha(x,y),\beta(x,y)))$ a.e. for $(x,y) \in \overline{D}_{x_0y_0}$, and we obtain $\dfrac{\partial^2 U(x,y)}{\partial x \partial y} = V(x,y)$ from (3.12); then, using (3.6) and (3.13) we have

$$V(x,y) = \frac{\partial^2 U(x,y)}{\partial x \partial y} \in F(x,y,Z(\alpha(x,y),\beta(x,y))), \quad \text{a.e. for } (x,y) \in \overline{D}_{x_0y_0}, \tag{3.14}$$

and also

$$U(x,y) = \varphi(x,y) \text{ for } (x,y) \in G_0 = P_0 - D_{x_0y_0}; \tag{3.15}$$

hence U satisfies initial conditions (1.2) for $(x,y) \in G_0$.

Let us take $\lambda \in C^*(\Delta; I\!R^n)$ with $\lambda(x,y) \in M$ for $(x,y) \in \Delta_0$. To each $Z \in \mathcal{Z}_M$ we associate the set $\Phi(Z) \subset \mathcal{Z}_M$ as follows:

$$(3.16)\quad U \in \Phi(Z) \Leftrightarrow U \in \mathcal{Z}_M,\ \frac{\partial^2 U(x,y)}{\partial x \partial y} \in F(x,y,Z(\alpha(x,y),\beta(x,y))),\ \text{a.e. } (x,y) \in \Delta_0.$$

We thus define a multifunction $\Phi : \mathcal{Z}_M \to 2^{\mathcal{Z}_M}$. The set $\Phi(Z)$ is *convex*, *compact* and *non-empty*. It can be seen that $\Phi(Z)$ is convex since $F(x,y,Z(x,y))$ is convex by hypothesis (H_1). We have $\Phi(Z) \subset \mathcal{Z}_M$ but $\mathcal{Z}_M$ is compact. The multifunction Φ has a closed graph because *graph* Φ is the set $\mathcal{G}$ for each fixed λ and $\mathcal{G}$ is closed. It follows that $\Phi(Z)$ is compact in $C(P_0; I\!R^n)$ as a closed subset of the compact set $\mathcal{Z}_M$. The set $\Phi(Z)$ is non-empty since there exists U, defined by (3.8) with the property $U \in \Phi(Z)$.

The multifunction $\Phi : \mathcal{Z}_M \to 2^{\mathcal{Z}_M}$ having a closed graph, is upper-semicontinuous by Theorem 2.1. Taking into account all the properties of Φ, the Kakutani-Ky Fan fixed point Theorem [10], [17] can be applied.

Indeed, $\Phi : \mathcal{Z}_M \to 2^{\mathcal{Z}_M}$ is defined on $\mathcal{Z}_M$ which is a convex, compact and non-empty set; it is also upper-semicontinuous and its set-values $\Phi(Z)$ are convex, closed and non-empty in $\mathcal{Z}_M$. From Kakutani-Ky Fan fixed point Theorem it follows that the multifunction Φ has at least a fixed point, i.e. there exists at least an element $Z \in \mathcal{Z}_M$ such that $Z \in \Phi(Z)$, hence $Z = U$; but U is of the form (3.8), therefore this fixed point Z is a solution of Goursat-Ionescu Problem (1.1)+(1.2).

ii) We denote by S_λ the set of solutions to Problem (1.1)+(1.2), a notation showing that any solution Z depends on the function λ defined by (3.1). The set S_λ contains at least one element. The set S_λ is *compact*, *non-empty* in the Banach space $C(P_0; I\!R^n)$, being the set of the fixed points of multifunction Φ.

iii) The graph $\mathcal{H}$ of the multifunction $\lambda \to S_\lambda$, defined on $C^*(\Delta_0; I\!R^n)$ with values in $2^{\mathcal{Z}_M}$, $S_\lambda \subset \Phi(\mathcal{Z}_M) \subset 2^{\mathcal{Z}_M}$, is closed in $C^*(\Delta_0; I\!R^n) \times \mathcal{Z}_M$ since $\mathcal{H}$ is the image of the compact set $\mathcal{H}_1$ of the triples $(\lambda, Z, U) \in \mathcal{G}$ with $Z = U$, through the projection mapping $(\lambda, Z, U) \to (\lambda, Z)$. The mapping $\lambda \to S_\lambda$ is $-$ in general $-$ a multifunction because several solutions of the Problem (1.1)+(1.2) can exist, which are fixed points of mapping Φ corresponding to the same function λ. Because the mapping $\lambda \to S_\lambda$ has a closed graph $\mathcal{H}$ by Theorem 2.1, it follows that $\lambda \to S_\lambda$ is upper-semicontinuous on $C^*(\Delta_0; I\!R^n)$, what completes the proof.

References

[1] Carathéodory C., *Vorlesungen über Reelle Funktionen*, Chelsea Publishing Company, New York, 1968, 3 Ed.

[2] Castaing Ch., Sur les équations différentielles multivoques, *Comptes Rendus Acad. Sci. Paris*, T. 263, No. 2 (1966), Série A, 63-66.

[3] Castaing Ch., Quelques problèmes de mésurabilité liés à la théorie de la commande, *Comptes Rendus Acad. Sci. Paris*, T. 262, No. 7 (1966), Série A, 409-411.

[4] Cinquini Cibrario M., Cinquini S., *Equazioni a derivate parziali di tipo iperbolico*, Ed. Cremonese, Roma, 1964.

[5] Deimling K., A Carathéodory theory for systems of integral equations, *Ann. Mat. Pura Appl.*, (4) 86 (1970), 217-260.

[6] Deimling K., Das Picard-Problem für $u_{xy} = f(x,y,u,u_x,u_y)$ unter Carathéodory-Voraussetzungen, *Math. Z.*, 114(1970), 303-312.

[7] Deimling K., Das Goursat-Problem fur $u_{xy} = f(x, y, u)$, Aequationes Mathematicae, Vol. 6, Fasc. 2/3 (1971), 303-312.
[8] Dezsö G., On the Problem Goursat-Ionescu, Seminarul Naţional "Realizări şi perspective în domeniul traductoarelor pentru echipamente de măsură", Institutul Politehnic Cluj-Napoca, 1986, Vol. II, 151-158.
[9] Dezsö G., Principii de punct fix şi aplicaţii în teoria ecuaţiilor hiperbolice cu argument modificat, Teză de doctorat, Facultatea de Matematică şi Informatică, Universitatea "Babeş-Bolyai", Cluj-Napoca, 1 Aprilie 2000 (with English Abstract).
[10] Edwards R. E., *Functional Analysis. Theory and Applications*, Holt, Rinehart and Winston, 1965.
[11] Goursat E., *Traité d'Analyse*, Tome III (1923), 123-124.
[12] Hille E., Phillips R. S., *Functional Analysis and Semigroups*, American Mathematical Society, Colloquium Publications, Vol. 31(1957), Fourth printing of Revised Edition, 1981.
[13] Ionescu D. V., Sur une classe d'équations fonctionelles, Thèse, Paris (1927).
[14] Marano S., Generalized Solutions of Partial Differential Inclusions Depending on a Parameter, *Rend. Accad. Naz. Sc. XL, Mem. Mat.*, 13(1989), 281-295.
[15] Marano S., Classical Solutions of Partial Differential Inclusions in Banach Spaces, *Appl. Anal.*, 42, no. 2(1991), 127-143.
[16] Marano S., Controllability of Partial Differential Inclusions Depending on a Parameter and Distributed Parameter Control Processes, *Le Matematiche*, Vol. XLV, Fasc. II(1990), 283-300.
[17] Rus I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Editura Dacia, Cluj-Napoca, 1979.
[18] Straburzyński J., Existence of solutions of the Goursat Problem for some functional-differential equations, *Demonstratio Mathematica*, Vol. XV, No. 4 (1982), 883-897.
[19] Teodoru G., Le problème de Goursat pour une équation aux dérivées partielles multivoque, *Mathematica*, Tome 28(51), No. 2(1986), 185-188.