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ON NONLINEAR PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. The purpose of this paper is to study a initial-boundary value problem generated by a climate model of Diaz and Hetzer.
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Introduction

We shall consider initial-boundary value problems for the equation

$$D_t u(t,x) - \sum_{j=1}^{n} D_j[f_j(t,x,u(t,x),\nabla u(t,x))] + f_0(t,x,u(t,x),\nabla u(t,x)) +$$

$$h(t,x,[H(u)](t,x)) = F(t,x), \quad (t,x) \in Q_T = (0,T) \times \Omega$$

where $\Omega \subset R^n$ is an unbounded domain with sufficiently smooth boundary, H is a linear continuous operator in $L^p(Q_T)$, the functions f_j, h satisfy the Carathéodory conditions and certain polynomial growth conditions. We shall show that the weak solutions of this problem can be obtained as the limit (as $k \to \infty$) similar problems, considered in $(0,T)\times\Omega_k$ where $\Omega_k \subset \Omega$ are bounded domains with sufficiently smooth boundary, having the property $\Omega_k \supset \Omega \cap B_k$ ($B_k = \{x \in R^n : |x| < k\}$). Similar statements were proved in [13] for more special equations. There will be also proved a uniqueness theorem and the boundedness of the solutions if some additional conditions are satisfied.

The problem was motivated by the climate model considered by J.I. Díaz and G. Hetzer [8] where this type of equation was considered on the unit sphere in R^3 (instead of Ω). Some qualitative properties were proved in [1] and [7] for the climate model. Functional partial differential equations arise also in population dynamics, plasticity, hysteresis (see, e.g., [2], [5], [12], [19]).

In [15] similar problem was considered for bounded Ω, where the equation contained a rapidly increasing term with respect to u and also discontinuous terms in u. It is not difficult to extend the results of the present paper to higher order parabolic equations, containing discontinuous terms with respect to the unknown function.

1. Existence theorems

Let $\Omega \subset R^n$ be an unbounded domain with sufficiently smooth boundary, $p > 2$. For any domain $\Omega_0 \subset R^n$ denote by $W^{1,p}(\Omega_0)$ the usual Sobolev space with the norm

$$\| u \| = \left[\int_{\Omega_0} (|\nabla u|^p + |u|^p) \right]^{1/p}.$$

Let V be a closed linear subspace of $W^{1,p}(\Omega_0)$ and denote by $L^p(0,T;V)$ the Banach space of the set of measurable functions $u : (0,T) \to V$ such that $\| u \|^p$ is integrable and define the norm by

$$\| u \|^p_{L^p(0,T;V)} = \int_0^T \| u(t) \|^p_V \, dt.$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^\star)$ where $1/p + 1/q = 1$ and $V^\star$ is the dual space of V (see [9], [11], [18]).

Let γ be a continuous weight function satisfying

$$\gamma(x) \geq c_1 > 0 \text{ and } \int_\Omega \frac{1}{\gamma^{2/(p-2)}} < \infty$$

with some constant c_1. Denote by $W^{1,p}_\gamma(\Omega)$ the space of functions having a finite norm

$$\| w \| = \left[\int_\Omega [|\nabla w|^p + \gamma |w|^p] \right]^{1/p}.$$

By Hölder's inequality it is easy to show that $W^{1,p}_\gamma(\Omega)$ is continuously imbedded into $L^2(\Omega)$. Let V^γ be a closed linear subspace of $W^{1,p}_\gamma(\Omega)$ and $X^\gamma_T = L^p(0,T;V^\gamma)$.

Let $\varphi \in C^\infty_0(R^n)$ be a fixed function having the properties

$$0 \leq \varphi(x) \leq 1, \quad \varphi(x) = 1 \text{ if } |x| \leq 1/2, \quad \varphi(x) = 0 \text{ if } |x| \geq 3/4$$

and define function φ_k by $\varphi_k(x) = \varphi(x/k)$.

Assume that

A V_k is a closed linear subspace of $W^{1,p}(\Omega_k)$ such that for any $w \in V^\gamma$, $(\varphi_k w) |_{\Omega_k} \in V_k$.

Further, there exist linear and continuous (extension) operators $L_k : V_k \to V^\gamma$ such that for any $w_k \in V_k$, $(L_k w_k) |_{\Omega_k} = w_k$, for any $w \in V^\gamma$, $(L_k \varphi_k w) |_{\Omega_k} = \varphi_k w$, the sequence $\| L_k \|$ is bounded.

Remark 1. It is easy to show that assumption **A** is satisfied e.g. in the followig special cases:

a/ $V^\gamma = W^{1,p}_{\gamma,0}(\Omega)$, $V_k = W^{1,p}_0(\Omega_k)$;

b/ $\partial\Omega$ is bounded, $\Omega_k = \Omega \cap B_k$, $V^\gamma = W^{1,p}_\gamma(\Omega)$ and $V_k = W^{1,p}(\Omega_k)$;

c/ $\partial\Omega \in C^1$ is bounded, $\Omega_k = \Omega \cap B_k$, $V^\gamma = W^{1,p}_{\gamma,0}(\Omega)$ and $V_k = \{v \in W^{1,p}(\Omega_k) : v|_{\partial\Omega} = 0\}$.

Define the operators M_k by $(M_k v)(t,x) = v(t,\cdot) |_{\Omega_k} (x)$, $v \in X^\gamma_T$. Then we have $M_k(\varphi_k v) \in X^k_T = L^p(0,T;V_k)$.

Similarly, define the operators N_k by $(N_k v)(t,x) = (L_k v(t,\cdot))(x)$, $v \in X_T^k$. Then $N_k : L^p(0,T;V_k) \to L^p(0,T;V^\gamma)$ are linear and continuous, their norms are bounded.

On the functions f_j we assume that

B (i) $f_j : Q_T \times R \times R^n \to R$ are measurable in $(t,x) \in Q_T$ and continuous in $\eta \in R, \zeta \in R^n$;

(ii) $|f_j(t,x,\eta,\zeta)| \le c_1[(\gamma(x))^{1/q}|\eta|^{p-1} + |\zeta|^{p-1} + |\eta|] + k_1(x)$, $j = 1,...,n$,

$|f_0(t,x,\eta,\zeta)| \le c_1[(\gamma(x))^{p-1}|\eta|^{p-1} + |\zeta|^{p-1} + |\eta|] + k_1(x)$ with some constant c_1 and a function $k_1 \in L^q(\Omega)$;

(iii) $\sum_{j=1}^n [f_j(t,x,\eta,\zeta) - f_j(t,x,\eta,\tilde{\zeta})](\zeta_j - \tilde{\zeta}_j) > 0$ if $\zeta \neq \tilde{\zeta}$;

(iv) $\sum_{j=1}^n f_j(t,x,\eta,\zeta)\zeta_j + f_0(t,x,\eta,\zeta)\eta \ge c_2[|\zeta|^p + (\gamma(x))^{p-1}|\eta|^p] - k_2(x)$ with some constant $c_2 > 0$ and $k_2 \in L^1(\Omega)$.

Remark 2. A simple example for f_j satisfying **B** is

$$f_j(t,x,\eta,\zeta) = a_j(t,x)\zeta_j|\zeta_j|^{p-2} \quad (j = 1,...,n),$$

$$f_0(t,x,\eta,\zeta) = (\gamma(x))^{p-1}\eta|\eta|^{p-2} + b_0(t,x)\eta,$$

where a_j, b_0 are measurable functions, satisfying $0 < c_0 \le a_j(t,x) \le c_0'$, $0 \le b_0(t,x) \le c_0'$ with some constants c_0, c_0'.

On function h we assume

C (i) $h(t,x,\theta)$ is measurable in (t,x) and continuous in θ.

(ii) $|h(t,x,\theta)| \le k_3(x)k_4(|\theta|)(\gamma(x))^{p-1}|\theta|^{p-1} + k_5(x)$

where $k_3 \in L^1(\Omega) \cap L^\infty(\Omega)$, $\int_\Omega |k_5|^q \frac{1}{\gamma^{p-1}} < \infty$ and k_4 is a continuous function, satisfying $\lim_\infty k_4 = 0$.

Finally, assume that

D $H : L^p_\gamma(Q_T) \to L^p_\gamma(Q_T)$ is a linear and continuous operator (in the L^p space with the weight function γ) such that for any compact $K \subset \Omega$ there is a compact $\tilde{K} \subset \Omega$ with the following property: the restriction of $H(u)$to $(0,t) \times K$ depends only on the restriction of u to $(0,t) \times \tilde{K}$ for all $t \in (0,T]$ and it is continuous as an operator $L^p_\gamma(Q_t) \to L^p_\gamma(Q_t)$ with the same bounds for all t.

Remark 3. The operator H may have e.g. one of the following forms:

$$[H(u)](t,x) = \int_0^t \beta_0(s,t,x)u(s,x)ds \text{ or } [H(u)](t,x) = u(\tau(t),x)$$

with some $\beta_0 \in L^\infty((0,T)\times Q_T)$ and a continuously differentiable function τ satisfying $\tau' > 0, 0 < \tau(t) \le t$.

Define operators $A, B : X_T^\gamma \to (X_T^\gamma)^\star$ and $A_k, B_k : X_T^k \to (X_T^k)^\star$ by

$$[A(u),v] = \int_0^T \langle A(u)(t), v(t)\rangle dt =$$

$$\int_0^T \left[\sum_{j=1}^n \int_\Omega f_j(t,x,u,\nabla u)D_j v dx + \int_\Omega f_0(t,x,u,\nabla u)v dx\right] dt,$$

$$[B(u),v] = \int_0^T \langle B(u)(t), v(t)\rangle dt = \int_0^T \left[\int_\Omega h(t,x,H(u)(t,x))v dx\right] dt, \quad u,v \in X_T^\gamma;$$

$$[A_k(u_k), v_k] = \int_0^T \langle A_k(u_k)(t), v_k(t)\rangle dt =$$

$$\int_0^T \left[\sum_{j=1}^n \int_{\Omega_k} f_j(t, x, u_k, \nabla u_k) D_j v_k dx + \int_{\Omega_k} f_0(t, x, u_k, \nabla u_k) v_k dx \right] dt,$$

$$[B_k(u_k), v_k] = \int_0^T \langle B_k(u_k)(t), v_k(t)\rangle dt = \int_0^T \left[\int_{\Omega_k} h(t, x, H(N_k u_k)(t, x)) v_k dx \right] dt,$$

$u_k, v_k \in X_T^k$.

Finally, define for $F \in (X_T^\gamma)^\star$ its "restriction" $F_k \in (X_T^k)^\star$ by

$$[F_k, v_k] = [F, N_k v_k], \quad v_k \in X_T^k.$$

Theorem 1.1. *Assume* **A** - **D**. *Then for any* $F \in (X_T^\gamma)^\star$, $u_0 \in V^\gamma$ *there exists* $u_k \in X_T^k$ *satisfying*

$$\frac{du_k}{dt} + (A_k + B_k)(u_k) = F_k, \quad \frac{du_k}{dt} \in L^q(0, T; V_k^\star) \tag{1.1}$$

$$u_k(0) = M_k(\varphi_k u_0), \quad k = 1, 2, ...$$

Further, there exist a subsequence (u_{k_l}) *of the sequence* (u_k) *and* $u \in X_T^\gamma$ *such that*

$$(N_{k_l} u_{k_l}) \to u \text{ weakly in } X_T^\gamma$$

and u *satisfies*

$$\frac{du}{dt} + (A + B)(u) = F, \quad \frac{du}{dt} \in (X_T^\gamma)^\star \tag{1.2}$$

$$u(0) = u_0.$$

Proof. The existence of solutions u_k of (1.1) follows from the fact that $(A_k + B_k) : X_T^k \to (X_T^k)^\star$ is bounded, demicontinuous, pseudomonotone with respect to

$$D(L) := \{u \in X_T^k : \frac{du}{dt} \in (X_T^k)^\star, u(0) = 0\}$$

and it is coercive (see, e.g., [14], [15]). Thus by a known existence theorem (see, e.g., [3]) there exists a solution of (1.1).

Applying both sides of (1.1) to u_k we find

$$\frac{1}{2} \| N_k u_k(t) \|^2_{L^2(\Omega)} - \frac{1}{2} \| u_0 \|^2_{L^2(\Omega)} + c_3 \| N_k u_k \|^p_{L^p(0,t;V^\gamma)} \le \tag{1.3}$$

$$[\| F \|_{(X_T^\gamma)^\star} + c_4] \| N_k u_k \|_{L^p(0,t;V^\gamma)} + c_5$$

for all $t \in [0, T]$ with some positive constants c_3, c_4, c_5. This inequality implies that

$$\| N_k u_k \|_{X_T^\gamma} \text{ is bounded.} \tag{1.4}$$

Hence

$$A_k(u_k), \quad B_k(u_k) \text{ are bounded in } L^q(0, T; V_k^\star). \tag{1.5}$$

Further,

$$(N_k u_k) \to u \text{ weakly in } X_T^\gamma \text{ for a subsequence} \tag{1.6}$$

with some $u \in X_T^\gamma$. Define the "extensions" $\hat{A}_k(u_k)$ by

$$[\hat{A}_k(u_k), v] = [A_k(u_k), M_k(\varphi_k v)], \quad v \in X_T^\gamma,$$

then $\| \hat{A}_k(u_k) \|_{(X_T^\gamma)^\star}$ is bounded. Consequently, for a subsequence

$$(\hat{A}_k(u_k)) \to w \text{ weakly in } (X_T^\gamma)^\star \tag{1.7}$$

with some $w \in (X_T^\gamma)^\star$.

Since by (1.1), (1.4), (1.5) $\| \frac{du_k}{dt} \|_{(X_T^k)^\star}$ is bounded, by using also (1.4) we can choose a subsequence of (u_k) such that for any bounded $\Omega_0 \subset \Omega$,

$$(N_k u_k) \to u \text{ in } L^p((0,T) \times \Omega_0) \text{ and} \tag{1.8}$$

$$(N_k u_k) \to u \text{ a.e. in } Q_T. \tag{1.9}$$

By (1.8) and assumption **D** for a suitable subsequence

$$H(N_k u_k) \to H(u) \text{ a.e. in } Q_T. \tag{1.10}$$

Since for an arbitrary $v \in X_T^\gamma$

$$(M_k(\varphi_k v)) \to v \text{ in the norm of } X_T^\gamma \text{ and}$$

$$[F_k, M_k(\varphi_k v)] = [F, N_k(M_k(\varphi_k v))] = [F, \varphi_k v],$$

applying (1.1) to $M_k(\varphi_k v)$ with an arbitrary fixed $v \in X_T^\gamma$, we obtain as $k \to \infty$

$$\frac{du}{dt} + w + B(u) = F, \quad \frac{du}{dt} \in (X_T^\gamma)^\star \tag{1.11}$$

$$u(0) = u_0$$

(see, e.g., [18]).

Now we prove $w = A(u)$. Apply (1.1) to $M_k(u_k - u)\zeta$ with arbitrary fixed $\zeta \in C_0^\infty(\Omega)$ having the properties $\zeta \geq 0$, $\zeta(x) = 1$ in a compact subset K of Ω. So we obtain for sufficiently large k

$$[D_t u_k - D_t u, M_k((u_k - u)\zeta)] + [D_t u, M_k((u_k - u)\zeta)] + \tag{1.12}$$

$$[A_k(u_k), M_k((u_k - u)\zeta)] + [B_k(u_k), M_k((u_k - u)\zeta)] = [F_k, M_k((u_k - u)\zeta)].$$

For the first term (for sufficiently large k) we have

$$[D_t u_k - D_t u, M_k((u_k - u)\zeta)] = 1/2 \int_0^T \left[\frac{d}{dt} \int_\Omega (u_k(t) - u(t))^2 \zeta dx \right] dt = \tag{1.13}$$

$$1/2 \int_\Omega (u_k(T) - u(T))^2 \zeta dx \geq 0.$$

Further, by (1.6)

$$\lim_{k \to \infty} [D_t u, M_k((u_k - u)\zeta)] = 0, \tag{1.14}$$

$$\lim_{k \to \infty} [F_k, M_k((u_k - u)\zeta)] = \lim_{k \to \infty} [F, N_k(M_k((u_k - u)\zeta))] = \lim_{k \to \infty} [F, (u_k - u)\zeta] = 0.$$

By **D**, Hölder's inequality and (1.8)

$$\lim_{k \to \infty} [B_k(u_k), M_k((u_k - u)\zeta)] = 0. \tag{1.15}$$

Thus (1.12) - (1.15) imply

$$\limsup_{k\to\infty}[A_k(u_k), M_k((u_k-u)\zeta)] \le 0. \tag{1.16}$$

Since by **D**, Hölder's inequality and (1.8)

$$\lim_{k\to\infty}\int_{Q_{T,k}} f_0(t,x,u_k,\nabla u_k)(u_k-u)\zeta dtdx = 0 \quad (\text{where } Q_{T,k}=(0,T)\times\Omega_k),$$

(1.16) implies

$$\limsup_{k\to\infty}\sum_{j=1}^n\int_{Q_{T,k}} f_j(t,x,u_k,\nabla u_k)D_j[(u_k-u)\zeta]dtdx \le 0. \tag{1.17}$$

By using arguments of [6] (see also [13]) we obtain from (1.17)

$$\nabla u_k \to \nabla u \text{ a.e. in } (0,T)\times K.$$

Since K can be chosen as any compact subset of Ω, we find

$$\nabla(N_k u_k) \to \nabla u \text{ a.e. in } Q_T.$$

Thus Vitali's theorem and Hölder's inequality imply (see, e.g. [6])

$$(\hat{A}_k(u_k)) \to A(u) \text{ weakly in } (X_T^\gamma)^\star$$

i.e. $w = A(u)$ which completes the proof of our theorem.

Remark 4. It follows from the above proof that if the solution of (1.2) is unique then also for the original sequence (u_k) of solutions to (1.1), $(N_k u_k)$ converges weakly in X_T^γ to the solution u of (1.2).

If some additional conditions are satisfied then one can prove the uniqueness of the solution.

Theorem 1.2. *Assume* **A** - **D** *and the following monotonicity condition is satisfied:*

$$\sum_{j=1}^n[f_j(t,x,\xi)-f_j(t,x,\tilde{\xi})](\xi_j-\tilde{\xi}_j)+ \tag{1.18}$$

$$[f_0(t,x,\xi)-f_0(t,x,\tilde{\xi})](\xi_0-\tilde{\xi}_0) \ge -c_0(\xi_0-\tilde{\xi}_0)^2$$

with some constant c_0. Further, there exists a conctant k_0 such that

$$|h(t,x,\theta)-h(t,x,\tilde{\theta})| \le k_0|\theta-\tilde{\theta}| \tag{1.19}$$

for any $(t,x)\in Q_T$ and $\theta,\tilde{\theta}\in R$. Finally, H is positive, i.e. $u\ge 0$ implies $H(u)\ge 0$.

Then the solution of (1.2) is unique.

Proof. Perform the subtitution $u = e^{ct}\tilde{u}$. Then (1.2) is equivalent with

$$\frac{d\tilde{u}}{dt} + (\tilde{A}+\tilde{B})(\tilde{u}) + c\tilde{u} = \tilde{F}, \quad \frac{d\tilde{u}}{dt}\in (X_T^\gamma)^\star$$

$$\tilde{u}(0) = u_0$$

where

$$[\tilde{A}(\tilde{u}), v] =$$

$$\int_0^T \left[\sum_{j=1}^n \int_\Omega e^{-ct} f_j(t,x,e^{ct}\tilde{u}, e^{ct}\nabla\tilde{u}) D_j v dx + \int_\Omega f_0(t,x,e^{ct}\tilde{u}, e^{ct}\nabla\tilde{u}) v dx \right] dt,$$

$$[\tilde{B}(\tilde{u}), v] = \int_0^T \left[\int_\Omega e^{-ct} h(t,x,H(e^{ct}\tilde{u})(t,x)) v dx \right] dt, \quad \tilde{F} = e^{-ct} F.$$

and for sufficiently large c we obtain that the solution of the above problem is unique because then (by (1.18), (1.19)) the operator $\tilde{u} \mapsto (\tilde{A} + \tilde{B})\tilde{u} + c\tilde{u}$ is monotone.

It is not difficult to prove an existence theorem for the interval $[0, \infty)$. Denote by X^γ_∞ and $(X^\gamma_\infty)^\star$ the set of functions $u : [0,\infty) \to V^\gamma$, $w : [0,\infty) \to (V^\gamma)^\star$, respectively, such that for any finite T, $u \in X^\gamma_T$, $w \in (X^\gamma_T)^\star$, respectively. Further, define $Q_\infty = (0,\infty) \times \Omega$ and let $L^p_{\gamma,loc}(Q_\infty)$ be the set of functions $v : Q_\infty \to R$ such that $v \in L^p_\gamma(Q_T)$ for arbitrary finite T.

Theorem 1.3. *Assume* **A**, *further assume that* $f_j : Q_\infty \times R^{n+1} \to R$, $h : Q_\infty \times R \to R$ *satisfy* **B** *and* **C** *for any finite* $T > 0$ *and* $H : L^p_{\gamma,loc}(Q_\infty) \to L^p_{\gamma,loc}(Q_\infty)$ *satisfies* **D** *for any finite* T. *Then for arbitrary* $F \in (X^\gamma_\infty)^\star$ *there exists* $u \in X^\gamma_\infty$ *such that* u *satisfies (1.2) for any finite* T.

2. Boundedness of the solutions

Theorem 2.1. *Assume that the conditions* **A** - **D** *are satisfied for any finite* T *such that the constants and functions are independent on* T. *Further, assume that* $\| F(t) \|_{V^\star}$ *is bounded,*

$$|h(t,x,\theta)|^q \le c_4^\star |\theta|^2 + k_4^\star(x) \tag{2.20}$$

with some constant $c_4^\star$ *and a function* $k_4^\star \in L^1(\Omega)$; *for any* $u \in X^\gamma_\infty$

$$\int_\Omega |H(u)|^2(t,x) dx \le c_5^\star \sup_{\tau \in [0,t]} \int_\Omega |u(\tau,x)|^2 dx \tag{2.21}$$

with some constant $c_5^\star$.

Then for the solution u *of the problem in* Q_∞, *the function*

$$y(t) = \int_\Omega |u(t,x)|^2 dx$$

is bounded in $(0,\infty)$ *and there exist positive numbers* c', c" *such that*

$$\int_{T_1}^{T_2} \| u(t) \|_V^p \, dt \le c'(T_2 - T_1) + c\text{"} \text{ for sufficiently large } T_1 < T_2. \tag{2.22}$$

Remark 5. The examples in Remark 3. satisfy (2.21).

Proof. Apply (1.2) to u and integrate the equality over (T_1, T_2) with respect to t then we obtain

$$\int_{T_1}^{T_2} \langle D_t u(t), u(t) \rangle dt + \int_{T_1}^{T_2} \langle (A+B)(u)(t), u(t) \rangle dt + \int_{T_1}^{T_2} \langle F(t), u(t) \rangle dt. \tag{2.23}$$

For the first term in (2.23) we have

$$\int_{T_1}^{T_2} \langle D_t u(t), u(t)\rangle dt = \frac{1}{2}[y(T_2) - y(T_1)]. \tag{2.24}$$

By the assumption **B** (iv)

$$\int_{T_1}^{T_2} \langle A(u)(t), u(t)\rangle dt \geq c_2 \int_{T_1}^{T_2} \| u(t) \|_{V^\gamma}^p - (T_2 - T_1) \int_\Omega k_2. \tag{2.25}$$

Further, by (2.20), (2.21), for arbitrary number $\varepsilon > 0$

$$|\langle B(u)(t), u(t)\rangle| \leq \frac{\varepsilon^p}{p} \| u(t) \|_{L^p(\Omega)}^p + \frac{1}{\varepsilon^q q} \int_\Omega |h(t, x, [H(u)](t, x))|^q dx \leq \tag{2.26}$$

$$\frac{\varepsilon^p}{p} \| u(t) \|_V^p + \frac{c_4^\star c_5^\star}{\varepsilon^q q} \sup_{\tau \in [0,t]} \int_\Omega |u(\tau, x)|^2 dx + c_6^\star$$

with some constant $c_6^\star$. Finally, for the right hand side of (2.23)

$$\left| \int_{T_1}^{T_2} \langle F(t), u(t)\rangle dt \right| \leq \frac{\varepsilon^p}{p} \int_{T_1}^{T_2} \| u(t) \|_V^p + \frac{1}{\varepsilon^q q} \int_{T_1}^{T_2} \| F(t) \|_{V^\star}^q dt. \tag{2.27}$$

Choosing sufficiently small $\varepsilon > 0$, we obtain from (2.23) - (2.27)

$$\frac{1}{2}[y(T_2) - y(T_1)] + \frac{c_2}{2} \int_{T_1}^{T_2} \| u(t) \|_{V^\gamma}^p dt \leq \tag{2.28}$$

$$c_0 \int_{T_1}^{T_2} \sup_{\tau \in [0,t]} y(\tau) dt + c_6^\star (T_2 - T_1)$$

with some constant c_0. Since

$$y(t) = \int_\Omega |u(t, x)|^2 dx \leq const \left[\int_\Omega |u(t, x)|^p \gamma(x) dx \right]^{2/p},$$

we obtain

$$\frac{1}{2}[y(T_2) - y(T_1)] + \tilde{c}_0 \int_{T_1}^{T_2} [y(t)]^{p/2} dt \leq \tag{2.29}$$

$$c_0 \int_{T_1}^{T_2} \sup_{\tau \in [0,t]} y(\tau) dt + c_6^\star (T_2 - T_1)$$

with some positive $\tilde{c}_0$. It is not difficult to show that the last inequality implies the boundedness of y if $p > 2$ which will imply (2.22) by (2.28). Assume that y is not bounded. Then for any (sufficiently large) M there exist $t_0 > 0$ and $t_1 \in (0, t_0]$ such that

$$y(t_1) = \sup_{[0,t_0]} y = M.$$

Since y is continuous, there is a $\delta > 0$ such that

$$y(t) > M - 1 \text{ if } t_1 - \delta \leq t \leq t_1.$$

Applying (2.29) with $T_2 = t_1$ and $T_1 = t_1 - \delta$ we find

$$\frac{1}{2}[y(t_1) - y(t_1 - \delta)] + \frac{c_2}{2}\delta(M-1)^{p/2} \leq c_0\delta M + c_6^\star\delta$$

where $y(t_1) - y(t_1 - \delta) \geq 0$. Consequently,

$$\frac{c_2}{2}(M-1)^{p/2} \leq c_0 M + c_6^\star$$

which is impossible if M is sufficiently large.

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