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NON-TRIVIAL C^*-ALGEBRAS GENERATED BY IDEMPOTENTS

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Abstract. Every homogeneous C^*-algebra corresponds to the algebraic fibre bundle. C^*-algebra is called non-trivial if the corresponding algebraic fibre bundle is non-trivial. All C^*-algebras generated by idempotents that studied before corresponds to the trivial algebraic fibre bundles. In the work was showed that non-trivial C^*-algebras of any dimension $n \geq 3$ can be generated by three idempotents. It is follows from here that we need to study the topology properties of C^*-algebras generated by the idempotents to describe such algebras.

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1. Preliminaries

In order to formulate the main problem of the paper, we'll represent some necessary information on the theory of fiber bundles [2], [5].

A fiber bundle (bundle) is a triple (E, B, p), where E B are topological spaces, $p : E \to P$ is a continuous surjection. Here E is a bundle space, B is a base space and p is a projection. The subspace $E_x = p^{-1}(x) \subset E, \quad x \in B$ is called a fiber over the point x. Let F be a topological space. An example of a bundle is the product-bundle $E = B \times F$.

The bundles (E^1, B, p_1) and (E^2, B, p_2) with the same base B and the same structure group G are called isomorphic if there exists a homeomorphism $\phi : E^1 \to E^2$, which maps the fiber E^1_x onto E^2_x and this mapping belongs to the group G.

A bundle which is isomorphic to a bundle product is called *trivial.* The bundle (E, B, p) is called locally trivial with a fiber F, if each point of B has a neighborhood U such that the bundle E over U is trivial. In this case all fibers are isomorphic to a typical fiber F. If the fiber F is the C^*-algebra $Mat(n) = C^{n\times n}$ of matrices of dimension n and if the structure group that operates in each fiber is a group $Aut(n)$ of automorphisms of the algebra, then the bundle is called *algebraic.*

Let us to remind that C^*-algebra is a Banach algebra with one additional identity for the norm: $\|a^*a\| = \|a\|^2$.

If all irreducible representations of C^*-algebra have the same dimension n then the algebra is called *homogeneous.* The number n is called dimension of the algebra. An example of homogeneous algebra is the algebra $C(M, C^{n\times n})$ of continuous matrix-functions of dimension n on the compact space M. In this example the space M is the space $Prim$ of primitive ideals of algebra in the appropriate topology.

The element a of the algebra A is called idempotent if $a^2 = a$.

It was shown in the paper [2] that every homogeneous C^*-algebra is corresponds to the algebraic fiber bundle over compact space. C^*-algebra that isomorphic to the algebra of all continuous matrix-functions on compact space is called trivial C^*-algebra. Every trivial C^*-algebra is corresponds to the algebraic fiber bundle.

It is usual that non-trivial C^*-algebra is isomorphic to a subalgebra of algebra of all continuous functions over compact space. It was shown in the paper [1] that the non-trivial C^*-algebra of all sections of algebraic fiber bundles over the sphere S^2 can be reproduced as a subalgebra of the algebra $C(D, C^{n\times n})$, where $D = z \in C : |z| < 1$ is the unit disk. Let V(z) be a continuous matrix-function on the circle with values in the unitarian matrices of dimension n.

We'll denote via B_V the algebra of continuous matrix-functions $a(z)$ on the unit disk $z : |z| \leq 1$, which satisfies the condition $a(z) = V^{-1}(z)a(1)V(z)$, if $|z| = 1$. It was shown in the [1] that every homogeneous algebra A with the set of primitive ideals $PrimA = S^2$ is isomorphic to one of the algebras B_V. The next theorem is the criterium for two algebras B_V and B_W to be trivial.

Statement 1.1. *([1]). The algebras B_V and B_W are isomorphic if and only if $indV - indW = ln, l \in Z$.*

2. The main results

The next question is the main problem of the work. Can some C^*-algebras be generated by idempotents? The history of the question was reproduced in the [4]. It was shown in the [1] that each C^*-algebra of dimension 2, which corresponds to the algebraic bundle over the sphere S^2, can be generated by three idempotents. It is follows from the results of the work [4] that every trivial C^*-algebra over the sphere S^2 can be generated by idempotents. It is very interesting for us to know the answer on the next question. Can the non-trivial C^*-algebra over the sphere S^2 of dimension $n \geq 3$ be generated by idempotents? The answer is true and it was proved in the work [6]. In this paper we'll find the minimal number of idempotent generators for the non-trivial C^*-algebras over the sphere S^2.

We will use the next classical statement.

Statement 2.1. *(Stone and Weierstrass). Let $C(X)$ be the circle of the continuous complex-valued functions over the compact space X with the topology that generated by the norm $\| f \| = max_{x\in X}|f(x)|$, $f \in C(X)$.*

Also let $C_0 \subseteq C(X)$ is the sub-circle , that contains all constants and distinguishes all points of X, i.e. for each two points $x_1 \in X, x_2 \in X, (x_1 \neq x_2)$, there exists the function $f \in C_0$, such that $f(x_1) \neq f(x_2)$. Let the sub-circle C_0 contains both functions $f(x)$ and $\overline{f(x)}$, i.e. if $f(x) \in C_0 \Rightarrow \overline{f(x)} \in C_0$.

Then $[C_0] = C(X)$, i.e. every continuous function on X is the limit of functions from C_0.

The next theorem is the important step to describe the class of the C^*-algebras, that can be generated by three idempotents. The algebras B_V will be generated by idempotents P_1, P_2, P_3 as Banach algebras. In the previous results the algebras was generated as C^*-algebras, i.e. by the idempotents P_i and P_i^*.

Theorem 2.1. *C^*-algebras B_V of dimension 3 with the matrix-functions $V = (z^{-m}, 1, 1), m = 1, 2$ can be generated by three idempotents.*

Proof. Let the E_{ij} denote the matrix that has 1 on (ij)-th place and other elements are equal to 0. Let us to see the next matrix-functions of dimension 3.

$$P_1 = \begin{vmatrix} \mu_1 & \mu_3 q_{+m} & \mu_2 \\ \mu_3 q_{-m}\mu_1 & \mu_3^2 & \mu_3 q_{-m}\mu_2 \\ (1-|z|)\mu_1 & (1-|z|)\mu_3 q_{+m} & \mu_2(1-|z|) \end{vmatrix}$$

$$P_2 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} + \frac{1}{2}\overline{\mu_2}(1-|z|) \\ 0 & 0 & 0 \end{vmatrix}$$

$$P_3 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

, where $\mu_3(z) = (\frac{1}{6} - |z|)$. The function $q_{\pm m} = 1$, if $|z| < \frac{1}{6}$ and $z^{\pm m}$, if $|z| \geq \frac{1}{6}$, $\mu_2(z) = z(1-|z|)$, $\mu_1(z) = 1 - \mu_2(z)(1-|z|) - \mu_3(z)^2$. It can be directly checked that P_1, P_2, P_3 are idempotents. The matrix-function $P_3(z)$ belongs to any algebra B_V. The matrix-functions $P_1(z), P_2(z)$ belong to the algebra B_V with the matrix-function $V(z) = diag\{z^{-m}, 1, 1\}, m = 1, 2$.

Let the symbol A denote the algebra generated by the idempotents, i.e the minimal Banach algebra that contains the idempotents P_1, P_2, P_3. Our goal is to show that all matrix units $E_{ij} \in A, 2 \leq i, j \leq 3$.

The next goal is to prove that $\forall f \in C(D/S^1)$ the matrix-functions $fE_{33} \in A$, where the symbol D denotes the disk $\{z : |z| \leq 1\}$, S^1 denotes the circle $\{z : |z| = 1\}$, D/S^1 denotes the factor-space that homeomorphic to the sphere S^2.

Lemma 2.1. *The algebra A contains all $E_{ij}, 2 \leq i, j \leq 3$.*

Proof. We'll denote via $X := E_{22}(1 + (\frac{1}{2} + \frac{1}{2}\overline{\mu_2}(1-|z|))) = P_2P_3$.

Let us to consider the following sequence $h_n(x) := 1 - (1-x)^n$. The sequence $h_n(X) \to E_{22}, n \to \infty$. It is follows from the fact that $\| \overline{\mu_2}(1-|z|) \| < 1 \Rightarrow \| \frac{1}{2} + \frac{1}{2}\overline{\mu_2}(1-|z|)) \| < 1$ on the space D/S^1. Next, $E_{32} = P_3 - E_{22}$, and we'll denote $Y := E_{33}(\frac{1}{2} + \frac{1}{2}\overline{\mu_2}(1-|z|)) = (P_3 - E_{22})(P_2 - E_{22})$. Then $h_n(Y) \to E_{33}, n \to \infty$. It is follows from the next inequality $\| \frac{1}{2} - \frac{1}{2}\overline{\mu_2}(1-|z|)) \| < 1$ on the space D/S^1. Next step, $E_{33} = E_{32}E_{23}$. It is follows from above that all matrix units $E_{ij} \in A, 2 \leq i, j \leq 3$. The lemma is proved.

Lemma 2.2. $\forall f \in C(D/S^1) \Rightarrow fE_{ij} \in A, (1 \leq i, j \leq 2)$.

Proof. We'll show that some matrix-functions belong to the A and then we'll use the Stone-Weierstrass theorem.

$E_{33}P_1E_{33} = \mu_2(1-|z|)E_{33} \in A$.

$2E_{32}(P_2 - E_{22} - E_{33}) = \overline{\mu_2}(1-|z|)E_{33} \in A$.

$E_{32}P_1E_{23} = \mu_3^2E_{33} \in A$.

The functions $\mu_2(1-|z|)$ and μ_3^2 district the points of D/S^1. This fact was proved in the paper [6].

Now we can use the Stone-Weierstrass theorem. It is follows from this theory that the functions $\mu_2(1-|z|)$, $\overline{\mu_2}(1-|z|)$, μ_3^2 and the identity generates the algebra of all continuous functions on the D/S^1. It is means that for any continuous function $f \in C(D/S^1) \Rightarrow fE_{33} \in A$. It is follows from above that $fE_{ij} \in A, 2 \leq i,j \leq 3$. Indeed, the matrix-function $fE_{ij} = E_{i3}fE_{33}E_{3j}$. The lemma is proved.

Let us to consider the set $B_{\pm m}$ – the set of continuous functions on the unit disk D that satisfies one additional condition on the circle S^1. $B_{\pm m} = \{u \in C(D)|u = z^{\pm m}u(1), |z| = 1\}$. The set $B_{\pm m}$ has the structure of module over the algebra $C(D/S^1)$. The axioms of module are satisfies. We omit the details.

Our next goal is to show that if $g_1 \in B_{+m}$, then $E_{13}g_1 \in A$, and also, if $g_2 \in B_{-m}$, then $E_{31}g_2 \in A$. In order to prove this fact we'll show that the matrix-functions $E_{13}\mu_2 \in A$ and $E_{13}\mu_3q_{+m} \in A$. Also, we can multiply these matrix-functions to the elements from $C(D/S^1)$. Let f_1 and f_2 denote any two functions from the $C(D/S^1)$.

$P_1f_1E_{33} - \mu_3q_{-1}\mu_2E_{32} - \mu_2(1-|z|)E_{33} = E_{13}f_1\mu_2 \in A$

$P_1f_2E_{32} - \mu_3^2E_{32} - \mu_2(1-|z|)E_{33} = E_{13}f_2\mu_3q_{+m} \in A$

Similar, $E_{31}\mu_1(1-|z|) \in A$ and $E_{31}\mu_1\mu_3q_{-m} \in A$.

$f_1E_{33}P_1 - (1-|z|)\mu_3q_{+m}E_{32} - \mu_2(1-|z|)E_{33} = E_{31}\mu_1(1-|z|)f_1 \in A$

$E_{32}(f_2E_{32}P_1 - \mu_3^2E_{22} - \mu_3q_{-m}\mu_2) = E_{31}\mu_1\mu_3q_{-m}f_2 \in A$

It can be directly checked that the function $\mu_1 \neq 0$ on the D/S^1. Consequently, the $(1-|z|)E_{31} \in A$ and the $\mu_3q_{-m}E_{31} \in A$.

We need to use the next lemma about the elements that generates $B_{\pm m}$. It is follows from the lemma that the set $E_{13}B_{+m} \in A$ and $E_{31}B_{-m} \in A$.

Lemma 2.3. *The module B_{+m} can be generated over the algebra $C(B/S^1)$ by the elements $\mu_3(z)q_{+m}$ and $\mu_2(z)$. The module B_{-m} can be generated over the algebra $C(B/S^1)$ by the elements $\mu_3(z)q_{-m}$ and $(1-|z|)$.*

Proof. Let the M_1 denote the submodule of B_{+m} generated by the elements μ_3q_{+m}. Let the M_2 denote the submodule of B_{+m} generated by μ_2. $M_1 = B_{+m}I_1$ and $M_2 = B_{+m}I_2$, where I_1 and I_2 are the ideals of algebra $C(B/S^1)$. The ideals of $C(B/S^1)$ can be defined by the points where the generation function has zero.The ideal $I_1 = \{u(z) \in C(B/S^1)|u(z) = 0, |z| = \frac{1}{6}\}$ and the ideal $I_2 = \{u(z) \in C(B/S^1)|u(z) = 0, z = 0 \text{ or } z = 1\}$. The sum of submodules $M_1+M_2 = B_{+m}(I_1+I_2)$.

It is enough to show that $I_1+I_2 = C(B/S^1)$ in order to prove that $M_1+M_2 = B_{+m}$. Let us to consider the next functions $||z| - \frac{1}{6}| \in I_1$ and $|z| - |z|^2 \in I_2$. The sum of these functions has no zeroes on the D/S^1. Then, $1 \in I_1 + I_2$ and consequently, $I_1 + I_2 = C(D/S^1)$. The first part of lemma is proved. The second part of lemma is very similar to the first one. In this case the ideals contains the ideals from the first part: $I_1' = I_1, I_2' \supset I_2$. Consequently, $I_1' + I_2' = C(B/S^1)$. The lemma is proved.

Our next goal is to show that if $f_1 \in C(D/S^1)$ then $f_1E_{11} \in A$. Let us to see two functions $f_2, f_3 \in C(D/S^1)$.

We have that $\mu_2(z)(1-|z|)E_{11} \in A$, because of

$\mu_2(z)(1-|z|)E_{11} = E_{13}\mu_2E_{31}(1-|z|)$. Similar, $\mu_3^2E_{11} \in A$, because of

$\mu_3^2 E_{11} = E_{13}\mu_3 q_{+m} E_{31}\mu_3 q_{-m}$. The matrix-function $\mu_2(z)(1-|z|)f_1 E_{11}$ also belong to the algebra A, because of
$\mu_2(1-|z|)f_1 E_{11} = E_{13}\mu_2 E_{33} f_1 E_{31}(1-|z|)$. Similar,
$\mu_3^2 f_2 E_{11} = E_{13}\mu_3 q_{+m} E_{33} f_2 E_{31}\mu_3 q_{-m}$

The functions $\mu_2(z)(1-|z|)$ and $\mu_3^2(z)$ generates the ideals J_1, J_2 in $C(D/S^1)$. The sum of the ideals $J_1 + J_2 = C(D/S^1)$, because of the functions $\mu_2(1-|z|)$ and μ_3^2 does not equal to zero in the same point.

The next lemma will finish the proof of the theorem 2.1.

Lemma 2.4. *The algebra B_V is the direct sum of the modules over the algebra $C(D/S^1)$.*

More precisely, $B_V = E_{11}C(D/S^1) + E_{12}B_{+m} + E_{13}B_{+m} + E_{21}B_{-m} + E_{31}B_{-m} + \sum_{i,j=2}^{3} E_{ij}C(D/S^1)$.

Proof. The main idea is to consider the elementary matrices from B_V. Any matrix-function $R \in B_V$ looks like this:

$$R = \begin{vmatrix} C(B/S^1) & B_{+m} & B_{+m} \\ B_{-m} & C(B/S^1) & C(B/S^1) \\ B_{-m} & C(B/S^1) & C(B/S^1) \end{vmatrix}$$

The matrix R has the appropriate elements from $B_{\pm m}$ and $C(D/S^1)$. The lemma is proved.

It is follows from the lemma 2.2 that $E_{ij}C(D/S^1) \in A, 2 \le i,j \le 3$. It is follows from the lemma 2.3 that
$E_{1j}B_{+m} \in A, E_{i1}B_{-m} \in A, 2 \le i,j \le 3$. And also, $E_{11}C(D/S^1) \in A$. Now we have that all necessary elements belongs to A. Now we can use the lemma 2.3, and as result we have that $B_V = A$. The theorem is proved.

We can generalize the results of the lemmas 2.2, 2.3, 2.4 in the next theorem.

Theorem 2.2. *Let A is denote the closed subalgebra of the algebra B_V. Let the matrix-function $V = (z^{-m}, 1, .., 1), 1 \le m \le n-1$. Let the elements $E_{1n}\mu_2 \in A$, $E_{1n}\mu_3 q_{+m} \in A$, $E_{n1}\mu_2 \in A$, $E_{n1}(1-|z|) \in A$.*

Let the matrix units $E_{ij} \in A, 2 \le i,j \le n$ and $\overline{\mu_2}(1-|z|)E_{nn} \in A$.

Then $A = B_V$.

Proof. The theorem in the case $n = 3$ was proved in the lemmas 2.2 – 2.4. The proofs of lemmas 2.2 – 2.4 can be simply modified for another dimensions $n \ge 2$. We omit the details. The theorem is proved.

The next theorem will find the minimal number of idempotent generators for the homogeneous algebras over the S^2 of dimension $n \ge 4$.

Theorem 2.3. *C^*-algebras B_V of dimension $n \ge 4$ with matrix-function $V = (z^{-m}, 1, .., 1), m = 1, .., n-1$ can be generated by three idempotents.*

Proof. We'll denote via A_1, A_2 the next matrix-functions of dimension 4.

$$A_1 = \begin{vmatrix} 1 & \mu_3 q_{+m} & 0 & \mu_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$A_2 = \begin{vmatrix} 1 & 0 & 0 & \mu_2 \\ 0 & 0 & \text{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

One can directly checks that A_1, A_2 are idempotents. The functions $\mu_2(z), \mu_3(z)$ and $q_{\pm m}$ are the same as in the theorem 1. There is only one difference: m has values between 1 and $n-1$.

We'll denote as B_l and C_l the next upper-triangular matrices of dimension l.

$$B_l = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 1 & 1 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 1 & . \\ . & . & . & . & . & \ddots \end{vmatrix}$$

$$C_l = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & \text{-1} & 0 & 0 & . \\ 0 & 0 & 1 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & \text{-1} & . \\ 0 & 0 & 0 & 0 & 1 & . \\ . & . & . & . & . & \ddots \end{vmatrix}$$

It can be directly checked that B_l and C_l are idempotents. Further, we'll define the matrix-functions P_1 and P_2 of dimension n. The P_1 is the block-diagonal matrix-function that has two blocks, A_1 and B_{n-4}. The P_2 has two blocks A_2 and C_{n-4}. In the case $n = 4$ the $P_1 = A_1$ and the $P_2 = A_2$.

The matrix-functions P_1 and P_2 are the direct sum of two blocks. All these blocks are idempotents. Therefore, P_1 and P_2 are idempotents.

The matrix-function P_3 has the following definition:

$$P_3 = \begin{vmatrix} 0 & 0 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 0 & 0 \\ \mu_3 q_{-m} & 0 & . & 1 & 0 \\ 1-|z| & \frac{1}{2}+\frac{1}{2}\overline{\mu_2} & . & 0 & 1 \end{vmatrix}$$

The last line has the units on the free space. All other values are equal to zero.

One can directly checks that the P_3 is idempotent.

The matrix function P_2 belong to any algebra B_V of dimension n. The matrix-functions P_1 and P_3 belong to the algebra B_V with the matrix $V = diag(z^{-m}, 1, .., 1)$.

Let us to denote via A the minimal Banach algebra that contains the idempotents P_1, P_2, P_3.

The proof that algebra $A = B_V$ is very similar to the proof of the theorem 2.1. This proof also using the result of the theorem 2.2. We omit the details.

The theorem is proved. It is follows from the results of [1] and [4] that the algebras B_V cannot be generated by two idempotents. Three is the minimal number of idempotent generators for the algebras B_V for any dimension $n \geq 3$.

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