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INTEGRAL INCLUSIONS OF FREDHOLM TYPE RELATIVE TO MULTIVALUED φ-CONTRACTIONS

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Abstract. The purpose of this paper is to give a data dependence theorem of the fixed points set of a multivalued φ-contraction, where φ is a strong comparison function.

We also prove an existence theorem for the integral inclusion of Fredholm type

$$x(t) \in \int_a^b K(t,s,x(s))\ ds + f(t),\ t \in [a,b],$$

where K is a multivalued operator which satisfies a condition where appears a strong comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and we give a data dependence theorem of the solutions set of this Fredholm integral inclusion, using the established data dependence theorem of the fixed points set for a multivalued φ-contraction.

Keywords: multivalued operator, strong comparison function, fixed point, data dependence, integral inclusion.

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1. Introduction

Let X be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of X. For a multivalued operator $T : X \to P(X)$ we denote by F_T the set of the fixed points of T, i. e.

$$F_T := \{\ x \mid x \in X,\ x \in T(x)\ \}.$$

If X is a linear space, then we put

$$P_{cv}(X) := \{\ Y \mid Y \in P(X) \text{ and } Y \text{ is a convex set }\}.$$

Let (X,d) be a metric space. We denote

$$P_{cl}(X) := \{\ Y \mid Y \in P(X) \text{ and } Y \text{ is a closed set }\},$$

$$P_{cp}(X) := \{\ Y \mid Y \in P(X) \text{ and } Y \text{ is a compact set }\}$$

and we consider the following functionals

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A,B) = \{\ d(a,b) \mid a \in A,\ b \in B\ \},$$

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H(A,B) = \left\{\ \sup_{a\in A} D(a,B),\ \sup_{b\in B} D(b,A)\ \right\}.$$

Definition 1.1. *We say that a function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a* strong comparison function *iff*

(i) φ *is strictly increasing,*
(ii) $\sum_{k=0}^{\infty} \varphi^k(t) < +\infty$, *for each* $t \in \mathbb{R}_+$.

Definition 1.2. *Let* (X, d) *be a metric space,* $T : X \to P(X)$ *be a multivalued operator and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a strong comparison function. We say that* T *is a* multivalued φ-contraction *iff*

$$H(T(x), T(y)) \leq \varphi(d(x, y)),$$

for each $x, y \in X$.

We denote by $\mathcal{C}([a, b], \mathbb{R}^n)$ the Banach space of the continuous functions $u : [a, b] \to \mathbb{R}^n$, equipped with Cebyshev's norm $\|u\|_C = \sup_{t\in[a,b]} \|u(t)\|$.

2. Data dependence of the fixed points set for multivalued φ-contractions

First of all we remind a result given by Węgrzyk in [10].

Theorem 2.1 (Węgrzyk [10])**.** *Let* (X, d) *be a complete metric space,* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a strong comparison function and* $T : X \to P_{cl}(X)$ *be a multivalued* φ*-contraction.*
Then $F_T \neq \emptyset$.

Remark 2.1. *It is not difficult to show that* F_T *is a closed set.*

We also remind the following well-known properties of the functional H (see [1], [3], [6]).

Lemma 2.1. *Let* (X, d) *be a metric space,* $A, B \in P(X)$ *and* $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. *Then for each* $a \in A$, *there exists* $b \in B$ *such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Lemma 2.2. *Let* (X, d) *be a metric space and* $A, B \in P(X)$. *If there exists* $\eta \in \mathbb{R}$, $\eta > 0$ *such that*

(i) *for each* $a \in A$, *there is* $b \in B$ *so that* $d(a, b) \leq \eta$,
(ii) *for each* $b \in B$, *there is* $a \in A$ *so that* $d(b, a) \leq \eta$,

then $H(A, B) \leq \eta$.

Further on we shall prove a data dependence theorem of the fixed points set for multivalued φ-contractions.

Theorem 2.2. *Let* (X, d) *be a complete metric space and* $T_1, T_2 : X \to P_{cl}(X)$ *be two multivalued operators. We suppose that*:

(i) *for each* $i \in \{1, 2\}$ *there exists* $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ *a strong comparison function such that* T_i *is multivalued* φ_i*-contraction*;
(ii) *there exists* $\eta > 0$ *such that*

$$H(T_1(x), T_2(x)) \leq \eta,$$

for each $x \in X$.

Then $F_{T_1}, F_{T_2} \in P_{cl}(X)$ *and*

$$H(F_{T_1}, F_{T_2}) \le \max \left\{ \inf_{u\in\mathbb{N}^*} s_1\left(\eta + \frac{1}{u}\right), \inf_{u\in\mathbb{N}^*} s_2\left(\eta + \frac{1}{u}\right) \right\},$$

where $s_i(t) = \sum_{k=0}^{\infty} \varphi_i^k(t)$, *for each* $t \in \mathbb{R}_+$, $i \in \{1, 2\}$.

Proof. The fact that $F_{T_1}, F_{T_2} \in P_{cl}(X)$ follows immediately from the Theorem 2.1 and the Remark 2.1.

In order to prove the second part of the conclusion we consider $i, j \in \{1, 2\}$, with $i \neq j$. Let $x_0 \in F_{T_i}$ and $u \in \mathbb{N}^*$. By Lemma 2.1 it follows that there exists $x_1 \in T_j(x_0)$ such that

$$d(x_0, x_1) \le H(T_i(x_0), T_j(x_0)) + 1/(2u) \le \eta + 1/(2u).$$

We put $t_0 = d(x_0, x_1) + 1/(2u)$. It is not difficult to see that we can choose $\varepsilon_1 > 0$ such that $0 < \varepsilon_1 < \varphi_j(t_0) - \varphi_j(d(x_0, x_1))$. Using the Lemma 2.1 it follows that there exists $x_2 \in T_j(x_1)$ such that

$$d(x_1, x_2) \le H(T_j(x_0), T_j(x_1)) + \varepsilon_1 \le \varphi_j(d(x_0, x_1)) + \varepsilon_1 < \varphi_j(t_0).$$

Now, we can choose $\varepsilon_2 > 0$ such that $0 < \varepsilon_2 < \varphi_j^2(t_0) - \varphi_j(d(x_1, x_2))$. Using again the Lemma 2.1 it follows that there exists $x_3 \in T_j(x_2)$ so that

$$d(x_2, x_3) \le H(T_j(x_1), T_j(x_2)) + \varepsilon_2 \le \varphi_j(d(x_1, x_2)) + \varepsilon_2 < \varphi_j^2(t_0).$$

By induction, we prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T_j, starting from x_0, such that

$$d(x_n, x_{n+1}) < \varphi_j^n(t_0),$$

for each $n \in \mathbb{N}$.

Using this property, we are able to write, for each $n \in \mathbb{N}$ and for every $p \in \mathbb{N}^*$, that

$$d(x_n, x_{n+p}) \le \sum_{k=n}^{n+p-1} d(x_k, x_{k+1}) < \sum_{k=n}^{n+p-1} \varphi_j^k(t_0) \le \sum_{k=n}^{\infty} \varphi_j^k(t_0). \tag{2.1}$$

This implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore a convergent sequence, because (X, d) is a complete metric space. Let $x^* = \lim_{n\to\infty} x_n$. For each $n \in \mathbb{N}$ we have

$$D(x^*, T_j(x^*)) \le d(x^*, x_{n+1}) + D(x_{n+1}, T_j(x^*)) \le d(x^*, x_{n+1}) + H(T_j(x_n), T_j(x^*)) \le$$
$$\le d(x^*, x_{n+1}) + \varphi_j(d(x_n, x^*)) \le d(x^*, x_{n+1}) + d(x_n, x^*).$$

Letting n to tend to infinity we get that $D(x^*, T_j(x^*)) = 0$, so $x^* \in T_j(x^*)$, i. e. $x^* \in F_{T_j}$. From the relation (2.1), letting p to tend to infinity, we obtain

$$d(x_n, x^*) \le \sum_{k=0}^{\infty} \varphi_j^k(t_0) = s_j(t_0),$$

for each $n \in \mathbb{N}$.

For $n = 0$ we have

$$d(x_0, x^*) \le s_j(t_0).$$

Taking into account the fact that $d(x_0, x_1) \leq \eta + 1/(2u)$, it follows that

$$d(x_0, x^*) \leq s_j(\eta + 1/u).$$

Hence

$$d(x_0, x^*) \leq \inf_{u \in \mathbb{N}^*} s_j(\eta + 1/u).$$

Now, using the Lemma 2.2, we obtain the conclusion of the theorem. ■

Remark 2.2. *Another version of the Theorem* 2.2 *was given by Sîntămărian in* [9].

Remark 2.3. *If in the Theorem* 2.2 *we take* $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ *defined as follows*: $\varphi_i(t) = a_i t$, *for each* $t \in \mathbb{R}_+$, *where* $a_i \in]0, 1[$, *for* $i \in \{1, 2\}$, *then we get*

$$H(F_{T_1}, F_{T_2}) \leq \eta \ / \ (1 - \max \{ \ a_1, \ a_2 \ \}).$$

For $a_1 = a_2 \in]0, 1[$ *this is a result given by Lim in* [4].

3. Integral inclusions of Fredholm type

The first result of this section is an existence theorem for an integral inclusion of Fredholm type.

Theorem 3.1. *We consider the integral inclusion of Fredholm type*

$$x(t) \in \int_a^b K(t, s, x(s)) \ ds + f(t), \ t \in [a, b] \tag{3.1}$$

and we suppose that:

(i) *the multivalued operator* $K : [a, b] \times [a, b] \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ *is such that for each* $x \in \mathcal{C}([a, b], \mathbb{R}^n)$, *the multivalued operator* $K_x(t, s) := K(t, s, x(s))$, $(t, s) \in [a, b] \times [a, b]$ *is lower semicontinuous*;

(ii) $f \in \mathcal{C}([a, b], \mathbb{R}^n)$;

(iii) *there exists* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a strong comparison function such that*

$$H(K(t, s, u_1), K(t, s, u_2)) \leq l(t, s) \ \varphi(\|u_1 - u_2\|),$$

for each $t, s \in [a, b]$ *and for every* $u_1, u_2 \in \mathbb{R}^n$, *where* $l(t, \cdot) \in L^1[a, b]$, *for each* $t \in [a, b]$ *and* $\sup_{t \in [a,b]} \int_a^b l(t, s) \ ds \leq 1$.

Then the integral inclusion (3.1) *has at least a solution in* $\mathcal{C}([a, b], \mathbb{R}^n)$.

Proof. We consider the multivalued operator $T : \mathcal{C}([a, b], \mathbb{R}^n) \to \mathcal{P}(\mathcal{C}([a, b], \mathbb{R}^n))$ defined as follows

$$T(x) = \left\{ v \in \mathcal{C}([a, b], \mathbb{R}^n) \middle| v(t) \in \int_a^b K(t, s, x(s)) \ ds + f(t), \ t \in [a, b] \right\},$$

for each $x \in \mathcal{C}([a, b], \mathbb{R}^n)$.

Let $x \in \mathcal{C}([a, b], \mathbb{R}^n)$. For the multivalued operator $K_x : [a, b] \times [a, b] \to P_{cp,cv}(\mathbb{R}^n)$, from Michael's selection theorem, we get that there exists a continuous operator $k_x : [a, b] \times [a, b] \to \mathbb{R}^n$ such that $k_x(t, s) \in K_x(t, s)$, for each $(t, s) \in [a, b] \times [a, b]$. It follows that $\int_a^b k_x(t, s) \ ds + f(t) \in T(x)$. So $T(x) \neq \emptyset$.

It is easy to show that $T(x)$ is a closed set.

Therefore, we are able to write that $T : \mathcal{C}([a, b], \mathbb{R}^n) \to P_{cl}(\mathcal{C}([a, b], \mathbb{R}^n))$.

Further on we shall prove that T is multivalued φ-contraction, i. e. $H_C(T(x_1), T(x_2)) \le \varphi(\|x_1 - x_2\|_C)$, for each $x_1, x_2 \in \mathcal{C}([a,b], \mathbb{R}^n)$, where by $H_C(\cdot,\cdot)$ we denoted the Pompeiu-Hausdorff distance corresponding to Cebyshev's norm.

Let $x_1, x_2 \in \mathcal{C}([a,b], \mathbb{R}^n)$ and $v_1 \in T(x_1)$. Then there exists $k_{x_1}(t,s) \in K_{x_1}(t,s)$, $(t,s) \in [a,b] \times [a,b]$ so that $v_1(t) = \int_a^b k_{x_1}(t,s)\; ds + f(t)$, $t \in [a,b]$.

But

$$H(K(t,s,x_1(s)), K(t,s,x_2(s))) \le l(t,s)\; \varphi(\|x_1(s) - x_2(s)\|),$$

for each $t, s \in [a,b]$. So, there exists $w(t,s) \in K_{x_2}(t,s)$ such that

$$\|k_{x_1}(t,s) - w(t,s)\| \le l(t,s)\; \varphi(\|x_1(s) - x_2(s)\|),$$

for each $(t,s) \in [a,b] \times [a,b]$.

Now, we can consider the multivalued operator U defined by

$$U(t,s) = K_{x_2}(t,s) \cap \{\; u \in \mathbb{R}^n \mid \|k_{x_1}(t,s) - u\| \le l(t,s)\; \varphi(\|x_1(s) - x_2(s)\|)\; \},$$

for each $(t,s) \in [a,b] \times [a,b]$. Taking into account the fact that the multivalued operator U is lower semicontinuous, it follows that there exists a continuous operator $k_{x_2} : [a,b] \times [a,b] \to \mathbb{R}^n$ such that $k_{x_2}(t,s) \in U(t,s)$, for each $(t,s) \in [a,b] \times [a,b]$. We have

$$v_2(t) = \int_a^b k_{x_2}(t,s)\; ds + f(t) \in \int_a^b K(t,s,x_2(s))\; ds + f(t),\;\; t \in [a,b]$$

and

$$\|v_1(t) - v_2(t)\| \le \int_a^b \|k_{x_1}(t,s) - k_{x_2}(t,s)\|\; ds \le$$

$$\le \int_a^b l(t,s)\; \varphi(\|x_1(s) - x_2(s)\|)\; ds \le \varphi(\|x_1 - x_2\|_C)\; \int_a^b l(t,s)\; ds,$$

for each $t \in [a,b]$.

So $\|v_1 - v_2\|_C \le \varphi(\|x_1 - x_2\|_C)$.

Interchanging the role of x_1 and x_2 and using the Lemma 2.2 we obtain that $H_C(T(x_1), T(x_2)) \le \varphi(\|x_1 - x_2\|_C)$, for each $x_1, x_2 \in \mathcal{C}([a,b], \mathbb{R}^n)$.

From the Theorem 2.1 we get that $F_T \ne \emptyset$. The conclusion of the theorem follows immediately, because it is clear that $S_{K,f} = F_T$, where by $S_{K,f}$ we denoted the solutions set of the integral inclusion (3.1). ■

Remark 3.1. *If in the Theorem* 3.1 *we take the function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *as follows:* $\varphi(t) = at$, *for each* $t \in \mathbb{R}_+$, *where* $a \in]0,1[$, *then we obtain a result given by Petruşel in* [5].

The next result is a data dependence theorem for the solutions set of some integral inclusions of Fredholm type.

Theorem 3.2. *We consider the integral inclusions of Fredholm type*

$$x(t) \in \int_a^b K_1(t,s,x(s))\; ds + f_1(t),\;\; t \in [a,b] \tag{3.2}$$

and

$$x(t) \in \int_a^b K_2(t,s,x(s))\ ds + f_2(t),\ t \in [a,b]. \tag{3.3}$$

We suppose that:

(i) *the multivalued operators* $K_1, K_2 : [a,b]\times[a,b]\times\mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ *are such that for each* $x \in \mathcal{C}([a,b],\mathbb{R}^n)$*, the multivalued operators* $K_{1x}(t,s) := K_1(t,s,x(s))$ *and* $K_{2x}(t,s) := K_2(t,s,x(s))$*,* $(t,s) \in [a,b]\times[a,b]$ *are lower semicontinuous;*

(ii) $f_1, f_2 \in \mathcal{C}([a,b],\mathbb{R}^n)$;

(iii) *for each* $i \in \{1,2\}$ *there exists* $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ *a strong comparison function such that*

$$H(K_i(t,s,u_1), K_i(t,s,u_2)) \leq l_i(t,s)\ \varphi_i(\|u_1 - u_2\|),$$

for each $t,s \in [a,b]$ *and for every* $u_1,u_2 \in \mathbb{R}^n$*, where* $l_i(t,\cdot) \in L^1[a,b]$*, for each* $t \in [a,b]$ *and* $\sup_{t\in[a,b]} \int_a^b l_i(t,s)\ ds \leq 1$*;*

(iv) *there exists* $\eta_K > 0$ *such that*

$$H(K_1(t,s,u), K_2(t,s,u)) \leq \eta_K,$$

for each $t,s \in [a,b]$ *and for every* $u \in \mathbb{R}^n$*.*

Then

$$H_C(S_{K_1,f_1}, S_{K_2,f_2}) \leq \max\ \left\{ \inf_{n\in\mathbb{N}^*}\ s_1\left(\eta + \frac{1}{n}\right), \inf_{n\in\mathbb{N}^*}\ s_2\left(\eta + \frac{1}{n}\right)\right\},$$

where S_{K_1,f_1} *is the solutions set of the integral inclusion* (3.2)*,* S_{K_2,f_2} *is the solutions set of the integral inclusion* (3.3)*,* $m = \max_{t\in[a,b]} \|f_1(t) - f_2(t)\|$*,* $\eta = \eta_K\ (b-a) + m$ *and* $s_i(t) = \sum_{n=0}^{\infty} \varphi_i^n(t)$*, for each* $t \in \mathbb{R}_+$*,* $i \in \{1,2\}$*.*

Proof. From the proof of the Theorem 3.1 it follows that for each $i \in \{1,2\}$ the multivalued operator $T_i : \mathcal{C}([a,b],\mathbb{R}^n) \to P_{cl}(\mathcal{C}([a,b],\mathbb{R}^n))$, defined by

$$T_i(x) = \left\{\ v \in \mathcal{C}([a,b],\mathbb{R}^n)\ \middle|\ v(t) \in \int_a^b K_i(t,s,x(s))\ ds + f_i(t),\ t \in [a,b]\ \right\},$$

for each $x \in \mathcal{C}([a,b],\mathbb{R}^n)$, is multivalued φ_i-contraction.

Further on we shall prove that $H_C(T_1(x), T_2(x)) \leq \eta$, for each $x \in \mathcal{C}([a,b],\mathbb{R}^n)$.

Let $x \in \mathcal{C}([a,b],\mathbb{R}^n)$ and $v_1 \in T_1(x)$. Then there exists $k_{1x}(t,s) \in K_{1x}(t,s)$, $(t,s) \in [a,b]\times[a,b]$ such that $v_1(t) = \int_a^b k_{1x}(t,s)\ ds + f_1(t)$, $t \in [a,b]$.

But

$$H(K_1(t,s,x(s)), K_2(t,s,x(s))) \leq \eta_K,$$

for each $t,s \in [a,b]$. So, there exists $w(t,s) \in K_{2x}(t,s)$ such that

$$\|k_{1x}(t,s) - w(t,s)\| \leq \eta_K,$$

for each $(t,s) \in [a,b]\times[a,b]$.

Now, we can consider the multivalued operator U defined by

$$U(t,s) = K_{2x}(t,s) \cap \{\ u \in \mathbb{R}^n\ |\ \|k_{1x}(t,s) - u\| \leq \eta_K\ \},$$

for each $(t,s) \in [a,b] \times [a,b]$. Taking into account the fact that the multivalued operator U is lower semicontinuous, it follows that there exists a continuous operator $k_{2x} : [a,b] \times [a,b] \to \mathbb{R}^n$ such that $k_{2x}(t,s) \in U(t,s)$, for each $(t,s) \in [a,b] \times [a,b]$. We have

$$v_2(t) = \int_a^b k_{2x}(t,s)\ ds + f_2(t) \in \int_a^b K_2(t,s,x(s))\ ds + f_2(t),\ t \in [a,b]$$

and

$$\|v_1(t) - v_2(t)\| \le \int_a^b \|k_{1x}(t,s) - k_{2x}(t,s)\|\ ds + \|f_1(t) - f_2(t)\| \le$$
$$\le \eta_K\ (b-a) + m = \eta,$$

for each $t \in [a,b]$.

Hence $\|v_1 - v_2\|_C \le \eta$.

Interchanging the role of T_1 and T_2 we get that $H_C(T_1(x), T_2(x)) \le \eta$, for each $x \in \mathcal{C}([a,b], \mathbb{R}^n)$.

Now, applying the Theorem 2.2 we obtain

$$H_C(F_{T_1}, F_{T_2}) \le \max\ \left\{ \inf_{n \in \mathbb{N}^*}\ s_1\left(\eta + \frac{1}{n}\right), \inf_{n \in \mathbb{N}^*}\ s_2\left(\eta + \frac{1}{n}\right) \right\}$$

and from this it follows the conclusion of the theorem, taking into account the fact that $S_{K_1,f_1} = F_{T_1}$ and $S_{K_2,f_2} = F_{T_2}$. ■

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