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SOME RESEARCH PERSPECTIVES IN NONLINEAR FUNCTIONAL ANALYSIS

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Abstract. The object of this lecture is to propose a series of conjectures and problems in different fields of analysis. They have been formulated with the aim of introducing some innovative methods in the study of classical topics, as open mappings, fixed points, critical points, global minima, control theory.
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We start recalling the following definition. Let $(E, \|\cdot\|)$ be a real normed space. A non-empty set $A \subset E$ is said to be *antiproximinal with respect to* $\|\cdot\|$ if, for every $x \in E \setminus A$ and every $y \in A$, one has

$$\|x - y\| > \inf_{z \in A} \|x - z\| \ .$$

CONJECTURE 1. - There exists a non-complete real normed space E with the following property: for every non-empty convex set $A \subset E$ which is antiproximinal with respect to each norm on E, the interior of the closure of A is non-empty.

The main reason for the study of Conjecture 1 is to give a contribution to open mapping theory in the setting of non-complete normed spaces. Actually, making use of Theorem 4 of [8], one can prove the following result.

THEOREM 1. - *Let X, E be two real vector spaces, C a non-empty convex subset of X, F a multifunction from C onto E, with non-empty values and convex graph.*

Then, for every non-empty convex set $A \subseteq C$ which is open with respect to the relativization to C of the strongest vector topology on X, the set $F(A)$ is antiproximinal with respect to each norm on E.

We now present

CONJECTURE 2. - Let E be a real Banach space, and let $J : E \to \mathbf{R}$ be a continuously Gâteaux differentiable functional. Assume that there are $r > 0$ and $x_0, x_1 \in E$, with $\|x_0 - x_1\| > r$, such that

$$\inf_{\|x-x_0\|=r} J(x) \geq \max\{J(x_0), J(x_1)\} .$$

Put

$$c = \inf_{u\in\mathcal{A}} \sup_{t\in[0,1]} J(u(t))$$

where $\mathcal{A}$ denotes the set of all continuous functions $u : [0,1] \to E$ such that $u(0) = x_0$, $u(1) = x_1$.

Then, for each $\epsilon > 0$, there exist an interval $I \subset [0,1]$, a $u \in \mathcal{A}$ and a continuous function $\varphi : I \times E \to \mathbf{R}$, with $\varphi(t, \cdot)$ ϵ-Lipschitzian in E for all $t \in I$, such that

$$\sup_{t\in I} |J(u(t)) - c| < \epsilon$$

and the set

$$\{(t,y) \in I \times E : J'(u(t))(y) = \varphi(t,y)\}$$

is disconnected.

CONJECTURE 3. - Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space, $X \subset H$ a non-empty compact convex set, $f : X \to X$ a continuous function different from the identity, ϵ a positive real number small enough. Denote by Λ_ϵ the set of all continuous function $\varphi : X \times H \to \mathbf{R}$ such that, for each $x \in X$, $\varphi(x, \cdot)$ is Lipschitzian in H, with Lipschitz constant less than or equal to ϵ. Consider Λ_ϵ equipped with the relativization of the strongest vector topology on the space $\mathbf{R}^{X\times H}$.

Then, the set

$$\{(\varphi, x, y) \in \Lambda_\epsilon \times X \times H : \langle f(x) - x, y\rangle = \varphi(x,y)\}$$

is disconnected.

Conjectures 2 and 3 have been formulated with the aim of finding drastically novel proofs of the mountain pass lemma and the Brower fixed point theorem, respectively. Actually, if they were true, then we could get the mentioned proofs by means of the following results from [15] (see also [12]):

THEOREM 2 ([15], Theorem 19). - *Let X be a connected topological space, E a real Banach space (with topological dual space E^*), A a continuous operator from X into E^*, φ a real function on $X \times E$ such that, for each $x \in X$, $\varphi(x, \cdot)$ is Lipschitzian in E, with Lipschitz constant $L(x) \geq 0$. Further, assume that the set*

$$\{(x,y) \in X \times E : A(x)(y) = \varphi(x,y)\}$$

is disconnected.

Then, there exists some $v \in X$ such that $\| A(v) \|_{E^*} \leq L(v)$.

THEOREM 3 ([15], Theorem 21). - *Let X be a connected and locally connected topological space, E a real Banach space, $A : X \to E^*$ a continuous operator with closed range. For each $\epsilon > 0$, denote by Λ_ϵ the set of all continuous functions φ :*

$X \times E \to \mathbf{R}$ such that, for each $x \in X$, $\varphi(x, \cdot)$ is Lipschitzian in E, with Lipschitz constant less than or equal to ϵ. Consider Λ_ϵ equipped with the relativization of the strongest vector topology on the space $\mathbf{R}^{X \times E}$, and assume that the set

$$\{(\varphi, x, y) \in \Lambda_\epsilon \times X \times E : A(x)(y) = \varphi(x, y)\}$$

is disconnected.

Then, A vanishes at some point of X.

We now come to a problem in control theory. Let a be a positive real number and let F be a given multifunction from $[0, a] \times \mathbf{R}^n$ into $\mathbf{R}^n$. We denote by $\mathcal{S}_F$ the set of all Carathéodory solutions of the problem $x' \in F(t, x), x(0) = 0$ in $[0, a]$. That is to say

$$\mathcal{S}_F = \{u \in AC([0, a], \mathbf{R}^n) : u'(t) \in F(t, u(t)) \quad \text{a.e. in } [0, a],\ u(0) = 0\}\ .$$

For each $t \in [0, a]$, put

$$\mathcal{A}_F(t) = \{u(t) : u \in \mathcal{S}_F\}.$$

In other words, $\mathcal{A}_F(t)$ denotes the attainable set at time t. Also, put

$$V_F = \bigcup_{t \in [0,a]} \mathcal{A}_F(t).$$

Finally, set

$$C_F = \{x \in \mathbf{R}^n : \{t \in [0, a] : x \in \mathcal{A}_F(t)\} \ \text{ is connected}\}.$$

PROBLEM 1. - Find conditions under which the set C_F is non-empty and open.

The study of the above problem would be interesting in view of the following result, where "dim" stands for covering dimension:

THEOREM 4 ([11], Theorem 9). - *Assume that F has non-empty compact convex values and bounded range. Moreover, assume that $F(\cdot, x)$ is measurable for each $x \in \mathbf{R}^n$ and that $F(t, \cdot)$ is upper semicontinuous for a.e. $t \in [0, a]$.*

Then, for every non-empty connected set $X \subseteq V_F \cap C_F$ which is open in $\mathrm{aff}(X)$ *and different from $\{0\}$, one has the following alternative:*
either

$$X \subseteq \mathcal{A}_F(a)$$

or

$$\dim(\mathcal{A}_F(t) \cap X) \geq \dim(X) - 1$$

for some $t \in]0, a[$.

Now, we are going to present a problem about a unusual way of finding global minima of functionals in Banach spaces. First, we recall the following result from [17] (see also [10], [13], [14], [18], [19]):

THEOREM 5 ([17], Theorem 2.1) . - *Let* $(T, \mathcal{F}, \mu)$ *be non-atomic measure space, with* $\mu(T) < +\infty$, E *a real Banach space, and* $f : E \to \mathbf{R}$ *a bounded below Borel functional such that*

$$\sup_{x \in E} \frac{f(x)}{\|x\|^{\gamma} + 1} < +\infty$$

for some $\gamma \in]0, 1[$.

Then, for every $p \geq 1$ *and every closed hyperplane* V *of* $L^p(T, E)$, *one has*

$$\inf_{u \in V} \int_T f(u(t)) d\mu = \inf_{u \in L^p(T,E)} \int_T f(u(t)) d\mu.$$

PROBLEM 2. - Let X be an infinite-dimensional real Banach space, and let $J : X \to \mathbf{R}$ be a bounded below functional satisfying

$$\sup_{u \in X} \frac{J(u)}{\|u\|^{\gamma} + 1} < +\infty \qquad (*)$$

for some $\gamma \in]0, 1[$.

Find conditions under which there exists a closed hyperplane V of X such that the restriction of J to V has a local minimum.

The motivation for the study of Problem 2 is as follows. Assume that we wish to minimize a bounded below Borel functional f on a real Banach space E satisfying the condition

$$\sup_{x \in E} \frac{f(x)}{\|x\|^{\gamma} + 1} < +\infty$$

for some $\gamma \in]0, 1[$.

For each $u \in L^1([0, 1], E)$, put

$$J(u) = \int_0^1 f(u(t)) dt \ .$$

So, J is bounded below and satisfies $(*)$ with $X = L^1([0, 1], E)$. Assume that there is some closed hyperplane V of $L^1([0, 1], E)$ such that the restriction of J to V has a local minimum, say u_0. By a result of Giner ([3]) u_0 is actually a global minimum of the restriction of J to V. On the other hand, by Theorem 5, we have

$$\inf_{u \in V} J(u) = \inf_{u \in L^1([0,1],E)} J(u)$$

and so u_0 is a global minimum of J in $L^1([0, 1], E)$. This easily implies that f has a global minimum in E.

Using a major tool adopted in the proof of Theorem 5, O. Naselli got the following wonderful characterization:

THEOREM 6 ([7], Theorem 1). - *Let* $(T, \mathcal{F}, \mu)$ *be a* σ*-finite non-atomic complete measure space,* X *a real topological vector space,* Φ *a linear homeomorphism from* X *onto* $L^1(T)$, *and* $f : T \times \mathbf{R} \to \mathbf{R}$ *a Carathéodory function.*

Then, the following are equivalent:

(a) *The set*

$$\{u \in X : f(t, \Phi(u)(t)) = 0 \quad a.e.\ in\ T\}$$

is arcwise connected and intersects each closed hyperplane of X.

(b) *The function*

$$t \to \inf\{|x| : x \in \mathbf{R},\ f(t,x) = 0\}$$

belongs to $L^1(T)$ *and, for almost every* $t \in T$*, one has*

$$\sup\{x \in \mathbf{R} : f(t,x) = 0\} = +\infty$$

and

$$\inf\{x \in \mathbf{R} : f(t,x) = 0\} = -\infty.$$

Surprisingly enough, the following problem seems to be open:

PROBLEM 3. Does Theorem 6 hold replacing $L^1(T)$ by $L^p(T)$ with $p > 1$?

The next problems concern a very particular function space introduced in [9]. Let m, n be two positive integers. Denote by $V(\mathbf{R}^n)$ the space of all functions $u \in C^\infty(\mathbf{R}^n)$ such that, for each bounded subset $\Omega \subset \mathbf{R}^n$, one has

$$\sup_{\alpha \in \mathbf{N}_0^n} \sup_{x \in \Omega} |D^\alpha u(x)| < +\infty,$$

where $D^\alpha u = \partial^{\alpha_1 + \ldots + \alpha_n} u / \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

PROBLEM 4. - Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that, for each $u \in V(\mathbf{R}^n)$, the composite function $x \to f(u(x))$ belongs to $V(\mathbf{R}^n)$.
Then, must f necessarily be of the form $f(t) = at + b$?

PROBLEM 5. - For each $\alpha \in \mathbf{N}_0^n$, with $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq m$, let $a_\alpha \in \mathbf{R}$ be given. Let $P : V(\mathbf{R}^n) \to V(\mathbf{R}^n)$ be the differential operator defined by putting

$$P(u) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha u$$

for all $u \in V(\mathbf{R}^n)$.
Find necessary and sufficient conditions in order that

$$P(V(\mathbf{R}^n)) = V(\mathbf{R}^n).$$

Up to date, the only (very partial) answer to Problem 5 is provided by the following

THEOREM 7 ([9], Theorem 4). - *Let* $a, b \in \mathbf{R} \setminus \{0\}$ *and* $h, k \in \mathbf{N}$. *For each* $u \in V(\mathbf{R}^2)$*, put*

$$P(u) = a\frac{\partial^h u}{\partial x^h} + b\frac{\partial^k u}{\partial y^k}.$$

Then, one has

$$P(V(\mathbf{R}^2)) = V(\mathbf{R}^2)$$

if and only if $|a| \neq |b|$.

The final problems we want to present come from specific applications to nonlinear boundary value problems of the following result obtained in [20].

THEOREM 8 ([20], Theorem 2.5). - *Let X be a non-empty sequentially weakly closed set in a reflexive real Banach space, and let $\Phi, \Psi : X \to]-\infty, +\infty]$ be two sequentially weakly lower semicontinuous functionals. Assume also that Ψ is (strongly) continuous. Denote by I the set of all real numbers $\rho > \inf_X \Psi$ such that $\Psi^{-1}(]-\infty, \rho[)$ is bounded and intersects the domain of Φ. Assume that $I \neq \emptyset$. For each $\rho \in I$, put*

$$\varphi(\rho) = \inf_{x \in \Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - \inf_{\overline{(\Psi^{-1}(]-\infty,\rho[))}_w} \Phi}{\rho - \Psi(x)} ,$$

where $\overline{(\Psi^{-1}(]-\infty, \rho[))}_w$ is the closure of $\Psi^{-1}(]-\infty, \rho[)$ in the relative weak topology of X. Furthermore, set

$$\gamma = \liminf_{\rho \to (\sup I)^-} \varphi(\rho)$$

and

$$\delta = \liminf_{\rho \to \left(\inf_X \Psi\right)^+} \varphi(\rho) .$$

Then, the following conclusions hold:

(a) For each $\rho \in I$ and each $\mu > \varphi(\rho)$, the restriction of the functional $\Phi + \mu\Psi$ to $\Psi^{-1}(]-\infty, \rho[)$ has a global minimum.

(b) If $\gamma < +\infty$, then, for each $\mu > \gamma$, the following alternative holds: either the restriction of $\Phi + \mu\Psi$ to $\Psi^{-1}(]-\infty, \sup I[)$ has a global minimum, or there exists a sequence $\{x_n\}$ of local minima of $\Phi + \mu\Psi$ such that $\Psi(x_n) < \sup I$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} \Psi(x_n) = \sup I$.

(c) If $\delta < +\infty$, then, for each $\mu > \delta$, there exists a sequence $\{x_n\}$ of local minima of $\Phi + \mu\Psi$, with $\lim_{n\to\infty} \Psi(x_n) = \inf_X \Psi$, which weakly converges to a global minimum of Ψ.

From now on, Ω is an open bounded subset of $\mathbf{R}^n$, with smooth boundary, and (for $p > 1$) $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ are the usual Sobolev spaces, with norms

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}$$

and

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx \right)^{\frac{1}{p}}$$

respectively.

Let $p > 1$, and let $: \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function.

Recall that a weak solution of the Dirichlet problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x,u) in \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx - \int_\Omega f(x, u(x)) v(x) dx = 0 \qquad (**)$$

for all $v \in W_0^{1,p}(\Omega)$. While, a weak solution of the Neumann problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2} \nabla u) = f(x,u) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

ν being the outer unit normal to $\partial\Omega$, is is any $u \in W^{1,p}(\Omega)$ satisfying identity $(**)$ for all $v \in W^{1,p}(\Omega)$.

Let us recall the following classical result by Ambrosetti and Rabinowitz:

THEOREM 9 ([1], Theorem 3.10). - *Assume that:*
(1) *there are two positive constants a, q, with $q < \frac{n+2}{n-2}$ if $n \geq 3$, such that*

$$|f(x,\xi)| \leq a(1 + |\xi|^q)$$

for all $(x, \xi) \in \Omega \times \mathbf{R}$;
(2) *there are constants $r \geq 0$ and $c > 2$ such that*

$$0 < c \int_0^\xi f(x,t) dt \leq \xi f(x, \xi)$$

for all $(x, \xi) \in \Omega \times \mathbf{R}$ with $|\xi| \geq r$;
(3) *one has*

$$\lim_{\xi \to 0} \frac{f(x,\xi)}{\xi} = 0$$

uniformly with respect to x.
Then, the problem

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega \\ u_{|\partial\Omega} = 0. \end{cases}$$

has a non zero weak solution.

What can be said if, in Theorem 9, condition (3) is removed at all ? Using Theorem 8 (part (a)), we got the following result:

THEOREM 10 ([21, Theorem 4). - *Assume that conditions* (1) *and* (2) *hold.*
Then, for each $\rho > 0$ and each μ satisfying

$$\mu > \inf_{u \in B_\rho} \frac{\sup_{v \in B_\rho} \int_\Omega \left(\int_0^{v(x)} f(x,\xi) d\xi \right) dx - \int_\Omega \left(\int_0^{u(x)} f(x,\xi) d\xi \right) dx}{\rho - \int_\Omega |\nabla u(x)|^2 dx}, \qquad (***)$$

where

$$B_\rho = \left\{ u \in W_0^{1,2}(\Omega) : \int_\Omega |\nabla u(x)|^2 dx < \rho \right\},$$

the problem

$$\begin{cases} -\Delta u = \frac{1}{2\mu} f(x,u) \text{ in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

has at least two weak solutions one of which lies in B_ρ.

The following problem naturally arises in connection with Theorems 9 and 10.

PROBLEM 6. - Under conditions (1) and (2), is there some $\rho > 0$ such that the infimum appearing in $(***)$ is less than $\frac{1}{2}$?

Clearly, if the answer to this problem was positive, then Theorem 10 would be a proper improvement of Theorem 9.

Again applying Theorem 8 (part (a)), in [23], we obtained the following bifurcation theorem:

THEOREM 11. - *Let $f, g : \Omega \times \mathbf{R} \to \mathbf{R}$ be two Carathéodory functions. Assume that:*
(i) *there is $s > 1$ such that*

$$\limsup_{\xi \to 0^+} \frac{\sup_{x\in\Omega} |f(x,\xi)|}{\xi^s} < +\infty \ ;$$

(ii) *there is $q \in]0,1[$ such that*

$$\limsup_{\xi \to 0^+} \frac{\sup_{x\in\Omega} |g(x,\xi)|}{\xi^q} < +\infty \ ;$$

(iii) *there are a non-empty open set $D \subseteq \Omega$ and a set $B \subseteq D$ of positive measure such that*

$$\limsup_{\xi \to 0^+} \frac{\inf_{x\in B} \int_0^\xi g(x,t)dt}{\xi^2} = +\infty \ , \ \liminf_{\xi \to 0^+} \frac{\inf_{x\in D} \int_0^\xi g(x,t)dt}{\xi^2} > -\infty \ .$$

Then, for some $\lambda^* > 0$ and for each $\lambda \in]0, \lambda^*[$, the problem

$$\begin{cases} -\Delta u = f(x,u) + \lambda g(x,u) \ \ in \ \Omega \\ u_{|\partial\Omega} = 0 \ , \end{cases} \tag{P_λ}$$

admits a non-zero, non-negative weak solution $u_\lambda \in C^1(\overline{\Omega})$. Moreover, one has

$$\limsup_{\lambda \to 0^+} \frac{\|u_\lambda\|_{C^1(\overline{\Omega})}}{\lambda^{\frac{q}{1-q}}} < +\infty$$

and the function

$$\lambda \to \frac{1}{2}\int_\Omega |\nabla u_\lambda(x)|^2 dx - \int_\Omega \left(\int_0^{u_\lambda(x)} f(x,\xi)d\xi \right) dx - \lambda \int_\Omega \left(\int_0^{u_\lambda(x)} g(x,\xi)d\xi \right) dx$$

is negative and decreasing in $]0, \lambda^*[$. If, in addition, f, g are continuous in $\Omega \times [0, +\infty[$ and

$$\liminf_{\xi \to 0^+} \frac{\inf_{x\in\Omega} g(x,\xi)}{\xi |\log \xi|^2} > -\infty \ ,$$

then u_λ is positive in Ω.

In view of [2], where problem (P_λ) is studied for particular nonlinearities, we point out the following problem:

PROBLEM 7. - Under the assumptions of Theorem 11, does problem (P_λ) admit a non-zero, non-negative, *minimal* solution for each $\lambda > 0$ small enough ?

In the two final theorems, λ denotes a function in $L^\infty(\Omega)$, with ess $\inf_\Omega \lambda > 0$. They have been established in [22] as applications of parts (b) and (c) of Theorem 8. Their non-smooth versions have been obtained in [4] and [6].

THEOREM 12 ([22], Theorem 3). - *Assume $p > n$. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, and $\{a_k\}$, $\{b_k\}$ two sequences in $\mathbf{R}^+$ satisfying*

$$a_k < b_k \ \ \forall k \in \mathbf{N}, \ \ \lim_{k\to\infty} b_k = +\infty, \ \ \lim_{k\to\infty} \frac{a_k}{b_k} = 0 \ ,$$

$$\max\left\{ \sup_{\xi\in[a_k,b_k]} \int_{a_k}^{\xi} f(t)dt, \ \sup_{\xi\in[-b_k,-a_k]} \int_{-a_k}^{\xi} f(t)dt \right\} \le 0 \ \ \forall k \in \mathbf{N}$$

and

$$\limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(t)dt}{|\xi|^p} = +\infty \ .$$

Then, for every $\alpha, \beta \in L^1(\Omega)$, with $\min\{\alpha(x), \beta(x)\} \ge 0$ a.e. in Ω and $\alpha \ne 0$, and for every continuous function $g : \mathbf{R} \to \mathbf{R}$ satisfying

$$\sup_{\xi\in\mathbf{R}} \int_0^\xi g(t)dt \le 0$$

and

$$\liminf_{|\xi|\to+\infty} \frac{\int_0^\xi g(t)}{|\xi|^p} > -\infty \ ,$$

the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

admits an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

THEOREM 13 ([22], Theorem 4). - *Assume $p > n$. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, and $\{a_k\}$, $\{b_k\}$ two sequences in $\mathbf{R}^+$ satisfying*

$$a_k < b_k \ \ \forall k \in \mathbf{N}, \ \ \lim_{k\to\infty} b_k = 0, \ \ \lim_{k\to\infty} \frac{a_k}{b_k} = 0 \ ,$$

$$\max\left\{ \sup_{\xi\in[a_k,b_k]} \int_{a_k}^{\xi} f(t)dt, \ \sup_{\xi\in[-b_k,-a_k]} \int_{-a_k}^{\xi} f(t)dt \right\} \le 0 \ \ \forall k \in \mathbf{N}$$

and

$$\limsup_{\xi\to 0} \frac{\int_0^\xi f(t)dt}{|\xi|^p} = +\infty \ .$$

Then, for every $\alpha, \beta \in L^1(\Omega)$, *with* $\min\{\alpha(x), \beta(x)\} \geq 0$ *a.e. in* Ω *and* $\alpha \neq 0$, *and for every continuous function* $g : \mathbf{R} \to \mathbf{R}$ *satisfying*

$$\sup_{\xi \in \mathbf{R}} \int_0^\xi g(t)dt \leq 0$$

and

$$\liminf_{\xi \to 0} \frac{\int_0^\xi g(t)}{|\xi|^p} > -\infty \ ,$$

the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

admits a sequence of non zero weak solutions which strongly converges to 0 *in* $W^{1,p}(\Omega)$.

A major problem about Theorems 12 and 13 is as follows:

PROBLEM 8. - In Theorems 12 and 13, when $g = 0$, are the conclusions still valid without the assumption

$$\lim_{k \to \infty} \frac{a_k}{b_k} = 0 \ ?$$

A partial answer to this problem (for Theorem 13) has recently been provided by G. Anello and G. Cordaro in [3].

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