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FIXED POINT THEORY WITH APPLICATIONS TO DYNAMICAL SYSTEMS AND FRACTALS

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Abstract. The first aim of this paper is to report some results in connection with some single-valued and multi-valued Caristi-type operators in complete metric spaces. Then, we will prove that each finite family of single-valued and multi-valued operators satisfying to some Meir-Keeler type conditions has a self-similar set.

Keywords: multi-valued operator, fixed point, Caristi-type operator, self-similar set.
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1. Introduction

Let (X, d) be a metric space and $P(X)$ be the space of all nonempty subsets of X. Denote by $P_p(X)$ the family of all nonempty subsets of X having the property "p", where "p" could be: b=bounded, cl=closed, cp=compact, etc.

Following Aubin and Siegel (see [3]), a set-valued dynamic system F on X is a multi-valued operator $F : X \to P(X)$. Any sequence $x_0, x_1, ..., x_n, x_{n+1}$,..., such that $x_{n+1} \in F(x_n)$, for each $n \in \mathbb{N}$ is called a dynamic process of F starting at x_0. The set $T(x_0) := \{x_n : x_{n+1} \in F(x_n), n \in \mathbb{N}\}$ is called the trajectory of this motion and the space X is the phase space. If Y is a nonempty subset of X and $F : Y \to P(X)$ is a multifunction, then by definition, an element $x \in Y$ is said to be:

i) a fixed point of F if and only if $x \in F(x)$.

ii) a strict fixed point of F if and only if $\{x\} = F(x)$.

We denote by $Fix(F)$ the set of all fixed points for F and by $SFix(F)$ the set of all strict fixed points for F. A fixed point for the multi-valued map F may be interpreted as a rest-point of the dynamic system while a strict fixed point for F can be regarded as an end-point of the system. Let us also remark that, if $f : Y \to X$ is a single-valued operator and we define $F : Y \to P(X)$ by $F(x) = \{f(x)\}$, then we get a (single-valued)dynamic system.

The study of set-valued dynamic systems has received more attention in the last twenty years. For example, Aubin-Siegel (see [3]), Justman (see [18]) and Tarafdar-Yuan (see [34]) established several existence and stability results for the strict fixed points of a set-valued dynamic system F, as well as some conditions that guarantee each dynamic process converges and its limit is a strict fixed point of F. In the same time, Maschler-Peleg(see [20]) investigated the stability of the so-called generalized nucleolars sets for set-valued dynamic systems and gave applications to game theory, while G.X.-Z. Yuan (see [37]), using some existence and uniqueness results and algorithms for the strict fixed points of a set-valued dynamic systems satisfying to some generalized contraction conditions, derived existence theorems of Pareto optima for mappings taking values in ordered Banach spaces.

On the other hand, Madelbrot introduced the notion of self-similar set in his book "The Fractal Geometry of Nature" in 1982, but Hutchinson and Hata developed independently the rigorous mathematical studies of self-similar sets in connections with the mathematics of fractals. (see [36] for more details) In few words, a self-similar set is a set consisting of retorts of itself. More precisely, let f_i, $i \in \{1, \ldots, m\}$ be continuous operators of X into itself. A nonempty compact set Y in X is, by definition, self-similar if it satisfies the condition $Y = \cup_{i=1}^{m} f_i(Y)$. Obviously, we may regard the above relation as a fixed point problem for an appropriate operator. More precisely, let $(P_{cp}(X), H)$ be the metric space of all nonempty compact subsets of X, where H denotes the Hausdorff-Pompeiu metric. If $T : (P_{cp}(X), H) \to (P_{cp}(X), H)$ is defined by $T(Y) = \bigcup_{i=1}^{m} f_i(Y)$, then the self-similar sets in X are the fixed points of T. If $X = \mathbb{R}^n$, it is well known that a self-similar set is a global attractor with respect to the dynamics generated by T in the phase set $P_{cp}(X)$ and its Hausdorff dimension is not, in general, an integer. For this reason, Y is a fractal and $P_{cp}(X)$ is the space of fractals. Moreover, self-similar sets among the fractals form an important class, since many of them have computable Hausdorff dimensions. (see [36] for more details) Moreover, if f_i are α-contractions for $i \in \{1, \ldots, m\}$ then the operator T is an α-contraction and hence has a unique fixed point. (see [36] for example) Also, when f_i are φ-contractions (where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, see [30]), I. A. Rus and independently Máté, showed that T is a φ-contraction too, having a unique fixed point. (see [33] and [21])

The purpose of this synthesis is to present several results joining these important fields: dynamic systems, mathematics of fractals and fixed point theory. More precisely, the first purpose of this paper is to report some results in connection with some single-valued and multi-valued Caristi-type operators in complete metric spaces. Then, we will prove that each finite family of single-valued and multi-valued operators satisfying to some Meir-Keeler type conditions has a self-similar set. We refer to [25], [26] and [27] for more details on these topics.

2. Fixed points and dynamical systems

Caristi's fixed point theorem states that each operator f from a complete metric space (X, d) into itself satisfing the condition:

there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+$ such that:

$$d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in X \tag{2.1}$$

has at least a fixed point $x^* \in X$, i. e. $x^* = f(x^*)$ (see [8]).

There are several extensions and generalizations of this important principle of the nonlinear analysis (see for example [5], [10], etc.).

One of the latest, asserts that if (X, d) is a complete metric space, $x_0 \in X$, $\varphi : X \to \mathbb{R}_+$ is lower semi-continuous and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\int_0^\infty \frac{ds}{1+h(s)} = \infty$, then each single-valued operator f from X to itself satisfying the condition:

$$\text{for each } x \in X, \frac{d(x, f(x))}{1 + h(d(x_0, x))} + \varphi(f(x)) \leq \varphi(x), \tag{2.2}$$

has at least a fixed point (see [39]).

For the multi-valued case, if F is an operator of the complete metric space X into the space of all nonempty subsets of X and there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+$ such that

$$\text{for each } x \in X, \text{ there is } y \in F(x) \text{ so that } d(x, y) + \varphi(y) \leq \varphi(x), \tag{2.3}$$

then the multi-valued map F has at least a fixed point $x^* \in X$, i. e. $x^* \in F(x^*)$.(see [24])

Moreover, if F satisfies the stronger condition:

$$\text{for each } x \in X \text{ and each } y \in F(x) \text{ we have } d(x, y) + \varphi(y) \leq \varphi(x), \tag{2.4}$$

then the multi-valued map F has at least a strict fixed point $x^* \in X$, i. e. $\{x^*\} = F(x^*)$.(see [1])

On the other hand, if F is a multi-valued operator with nonempty closed values and $\varphi : X \to \mathbb{R}_+$ is a lower semi-continuous function such that the following condition holds:

$$\inf \{\ d(x, y) + \varphi(y)\ :\ y \in F(x)\ \} \leq \varphi(x), \text{ for each } x \in X, \tag{2.5}$$

then F has at least a fixed point.(see [14])

It is easy to see that (2.4) $\Rightarrow$ (2.3) $\Rightarrow$ (2.5) and (2.5) $\Rightarrow$ (2.3) provided that F has nonempty compact values.

The purpose of this section is to present several new results in connection with the above mentioned single-valued and multi-valued Caristi-type operators in complete metric spaces.

Let (X, d) be a metric space. We consider the following functionals:

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A, B) = \inf \{\ d(a, b) \mid a \in A,\ b \in B\ \}$$

$$H : P_b(X) \times P_b(X) \to \mathbb{R}_+,\ H(A, B) = \max \{\ \sup_{a \in A} D(a, B),\ \sup_{b \in B} D(b, A)\ \}.$$

It is well-known that if (X, d) is a complete metric space, then $(P_{b,cl}(X), H)$ is also a complete metric space.

Definition 2.1. *Let (X, d) be a metric space and $F : X \to P(X)$ be a multi-valued map. A function $\varphi : X \to \mathbb{R}_+$ is called*

*(*i*) a* weak entropy *of F if the condition* (2.3) *holds*;

*(*ii*) an* entropy *of F if the condition* (2.4) *holds.*

The map $F : X \to P(X)$ is said to be *weakly dissipative* iff there exists a weak entropy of F and F is said to be *dissipative* iff there is an entropy of it.

Definition 2.2. *Let (X,d) be a metric space and $F : X \to P_{b,cl}(X)$ be a multi-valued operator. Then F is said to be*:

(i) a-contraction *iff there exists $a \in [0,1[$ such that*

$$H(F(x),F(y)) \leq a\ d(x,y),\ \textit{for each } x,y \in X;$$

(ii) Reich-type operator *iff there exist $a,b,c \in \mathbb{R}_+$, with $a+b+c<1$ such that*

$$H(F(x),F(y)) \leq a\ d(x,y) + b\ D(x,F(x)) + c\ D(y,F(y)),\ \textit{for each } x,y \in X.$$

Let us remark now, that if f is a (single-valued) a-contraction in a complete metric space X, then f satisfies condition (2.1) with $\varphi(x) = (1-a)^{-1}\ d(x,f(x))$, for each $x \in X$, so that part of the Banach contraction principle which says about the existence of a fixed point can be obtained by Caristi's theorem. For the multi-valued case we have the following result:

Theorem 2.1. *Let (X,d) be a complete metric space and $F : X \to P_{cp}(X)$ be an a-contraction $(0<a<1)$. Then*:

a) F satisfies the condition (2.4) with $\varphi(x) = (1-a)^{-1}\ D(x,F(x))$, for each $x \in X$.

b) If, in addition $F(x) \in P_{cp}(X)$, for each $x \in X$, then F is weakly dissipative with a weak entropy given by the formula $\varphi(x) = (1-a)^{-1}\ D(x,F(x))$, for each $x \in X$.

Proof. *a*) is Corrolary 1 in [14] and *b*) follows immediately from *a*) and the conditions (2.3) ⇔ (2.4). ■

Remark 2.1. *It is an open question if a multi-valued a-contraction $(0<a<1)$ is dissipative.*

First main result of this paper is the following:

Theorem 2.2. *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ be a Reich-type multi-valued map. Then there exists $f : X \to X$ a selection of F satisfying the Caristi-type condition (2.1).*

Proof. Let $\varepsilon > 0$ such that $a < \varepsilon < 1-b-c$. We denote by $U_x = \{\ y \in F(x) \mid \varepsilon\ d(x,y) \leq (1-b-c)\ D(x,F(x))\ \}$, for each $x \in X$. Obviously, for each $x \in X$, the set U_x is nonempty (otherwise, if $x \in X$ is not a fixed point of F and we suppose that for each $y \in F(x)$ we have $\varepsilon\ d(x,y) > (1-b-c)\ D(x,F(x))$, then we reach the contradiction $\varepsilon\ D(x,F(x)) \geq (1-b-c)\ D(x,F(x))$; if $x \in X$ is a fixed point of F, then clearly $U_x \neq \emptyset$).

We can choose a single-valued mapping $f : X \to X$ such that $f(x) \in U_x$, i. e. $f(x) \in F(x)$ and $\varepsilon\ d(x,f(x)) \leq (1-b-c)\ D(x,F(x))$, for each $x \in X$.

Then the following relations hold:

$$D(f(x),F(f(x))) \leq H(F(x),F(f(x))) \leq$$
$$\leq a\ d(x,f(x)) + b\ D(x,F(x)) + c\ D(f(x),F(f(x)))$$

and hence

$$(1-c)\ D(f(x),F(f(x))) - b\ D(x,F(x)) \leq a\ d(x,f(x)).$$

In view of this we obtain:

$$d(x, f(x)) = (\varepsilon - a)^{-1} \ [\varepsilon \ d(x, f(x)) - a \ d(x, f(x))] \leq$$
$$\leq (\varepsilon - a)^{-1} \ [(1 - b - c) \ D(x, F(x)) - (1 - c) \ D(f(x), F(f(x))) + b \ D(x, F(x))] =$$
$$= (1 - c)/(\varepsilon - a) \ [D(x, F(x)) - D(f(x), F(f(x)))].$$

If we define $\varphi : X \to \mathbb{R}_+$ by $\varphi(x) = (1 - c)/(\varepsilon - a) \ D(x, F(x))$, for each $x \in X$, then it is easy to see that

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \text{ for each } x \in X.$$

Moreover,

$$|\varphi(x) - \varphi(y)| = (1 - c)/(\varepsilon - a) \ |D(x, F(x)) - D(y, F(y))| \leq$$
$$\leq (1 - c)/(\varepsilon - a) \ |d(x, y) + H(F(x), F(y))| \leq$$
$$\leq (1 - c)/(\varepsilon - a) \ [d(x, y) + a \ d(x, y) + b \ D(x, F(x)) + c \ D(y, F(y))] =$$
$$= (1-c)(1+a)/(\varepsilon-a) \ d(x, y) + b(1-c)/(\varepsilon-a) \ D(x, F(x)) + c(1-c)/(\varepsilon-a) \ D(y, F(y)),$$

proving the fact that the function φ is a kind of single-valued Reich-type operator. ■

Remark 2.2. *If the multi-valued operator $F : X \to P_{cl}(X)$ is an upper semicontinuous Reich-type operator, then φ is a lower semicontinuous entropy of f (because the map $x \mapsto D(x, F(x))$ is lower semicontinuous).*

Remark 2.3. *If in Theorem 2.2 we take $b = c = 0$, then we obtain Theorem 5 in [17]. Moreover, we get that a multivalued a-contraction $(0 \leq a < 1)$ is weakly dissipative.*

Finally, we present the following generalization of L. van Hot result (see [14] or the introduction of this section):

Theorem 2.3. *Let (X, d) be a complete metric space, $x_0 \in X$, $\varphi : X \to \mathbb{R}_+$ be a lower semi-continuous and $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, non-decreasing function such that $\int_0^\infty \frac{ds}{1+h(s)} = \infty$. Consider the multi-valued operator $F :\to P_{cl}(X)$ satisfying the condition:*

$$\text{(2.6)} \qquad \textit{for each } x \in X, \quad \inf \{ \ \frac{d(x, y)}{1 + h(d(x_0, x))} + \varphi(y) : y \in F(x)\} \leq \varphi(x)$$

Then, the multi-valued operator F has at least a fixed point.

3. Fixed points and fractals

We start this section by listing some contractive-type conditions. If $f : X \to X$ is an operator let us consider the following conditions:

i) α-contraction condition:

(1) there is $\alpha \in [0, 1[$ such that for $x, y \in X$ we have $d(f(x), f(y)) \leq \alpha d(x, y)$

ii) ϵ-locally contractive condition (where $\epsilon > 0$)

(2) there is $\alpha \in [0, 1[$ such that for $x, y \in X$, $d(x, y) < \epsilon$ we have $d(f(x), f(y)) \leq \alpha d(x, y)$

iii) strict contraction condition:

(3) $x, y \in X, \ x \neq y$ we have $d(f(x), f(y)) < d(x, y)$

iv) Meir-Keeler-type condition:

(4) for each $\eta > 0$ there exists $\delta > 0$ such that $x, y \in X$, $\eta \leq d(x,y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$

v) ϵ-locally Meir-Keeler type condition (where $\epsilon > 0$)

(5) for each $0 < \eta < \epsilon$ there is $\delta > 0$ such that $x, y \in X$, $\eta \leq d(x,y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$.

Let us observe that, condition (1) implies (2) and (4), (4) implies (5) and each of the conditions (1) and (4) implies (3). Also, (2) implies (5). For other contractive-type conditions and the relations between them, we refer to [17], [23], etc.

A metric space (X, d) is said to be ϵ-chainable (where $\epsilon > 0$ is fixed) if and only if given $a, b \in X$ there is an ϵ-chain from a to b, that is a finite set of points $x_0, x_1, \ldots, x_n$ in X such that $x_0 = a$, $x_n = b$ and $d(x_{i-1}, x_i) < \epsilon$, for all $i \in \{1, 2, \ldots, n\}$. If $f : X \to X$ is a single-valued operator then $x^* \in X$ is a fixed point for f if and only if $x^* = f(x^*)$. We will denote by $Fixf$ the fixed points set of f.

An useful result is the following (see [32] for example) :

Lemma 3.1. *Let $A, B \in P_{cp}(X)$. Then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Let us consider now two fixed point principles given by Meir-Keeler [22] and Xu [35], that are needed in the proofs of the main results.

Theorem 3.1. *Let (X, d) be a complete metric space and f an operator from X into itself. If f satisfies the Meir-Keeler type condition (4) then f has a unique fixed point, i.e. $Fixf = \{x^*\}$. Moreover for any $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = x^*$.*

Theorem 3.2. *Let (X, d) be a complete ϵ-chainable metric space and $f : X \to X$ be an operator satisfying the ϵ-locally Meir-Keeler type condition (5). Then f has a fixed point.*

Let $f_i : X \to X$, $i \in \{1, \ldots, m\}$ be a finite family of continuous operators. Let us define $T_f : (P_{cp}(X), H) \to (P_{cp}(X), H)$ by

$$T_f(Y) = \cup_{i=1}^{m} f_i(Y). \tag{3.7}$$

The operator T_f is the so-called Barnsley-Hutchinson operator or the fractal operator generated by the system $f = (f_1, f_2, ...f_m)$. (see [4] and [37])

First main result of this section is:

Theorem 3.3. *Let (X, d) be a complete metric space and $f_i : X \to X$, for $i \in \{1, 2, \ldots, m\}$ are operators satisfying the Meir-Keeler type condition (4). Then the fractal operator $T_f : (P_{cp}(X), H) \to (P_{cp}(X), H)$ defined by the relation (3.7) is a Meir-Keller type operator, $FixT_f = \{A^*\}$ and $(T_f^n(A))_{n\in\mathbb{N}}$ converges to A^*, for each $A \in P_{cp}(X)$*

Proof. We shall prove that for each $\eta > 0$ there is $\delta > 0$ such that the following implication holds

$$\eta \leq H(A, B) < \eta + \delta \text{ we have } H(T_f(A), T_f(B)) < \eta.$$

Let us consider $A, B \in P_{cp}(X)$ such that $\eta \leq H(A, B) < \eta + \delta$.

If $u \in T_f(A)$ then there exists $j \in \{1, \ldots, m\}$ and $x \in A$ such that $u = f_j(x)$.

For $x \in A$ we can choose (see Lemma 2.1) $y \in B$ such that $d(x, y) \leq H(A, B) < \eta + \delta$. We have the following alternative:

If $d(x, y) \geq \eta$ then $\eta \leq d(x, y) < \eta + \delta$ implies $d(f_j(x), f_j(y)) < \eta$. Hence $D(u, T_f(B)) \leq d(u, f_j(y)) < \eta$.

On the other hand, if $d(x, y) < \eta$ then from (4) we have $d(f_j(x), f_j(y)) < d(x, y) < \eta$ and again the conclusion $D(u, T_f(B)) < \eta$.

Because $T_f(A)$ is compact we have that $\rho(T_f(A), T_f(B)) < \eta$.

Interchanging the roles of $T_f(A)$ and $T_f(B)$ we obtain $\rho(T_f(B), T_f(A)) < \eta$ and hence $H(T_f(A), T_f(B)) < \eta$, showing the fact that T_f is a Meir-Keeler-type operator. From Meir-Keeler fixed point result (Theorem 3.1 below) we obtain that there exists an unique $A^* \in P_{cp}(X)$ such that $T_f(A^*) = A^*$ and $(T_f^n(A))_{n\in\mathbb{N}}$ converges to A^*, for each $A \in P_{cp}(X)$. □

Remark 3.1. *By definition, the set A^* is called the attractor of the system $f = (f_1, f_2, ..., f_m)$. Hence, Theorem 3.3. is an existence result of an attractor.(see also [9], [15], [26], [33], etc.)*

Next we will prove a local version of the previous result:

Theorem 3.4. *Let (X, d) be a complete ϵ-chainable metric space and $f_i : X \to X$, for $i \in \{1, \ldots, m\}$ be operators satisfying the ϵ-locally-Meir-Keeler type condition (5). Then the fractal operator T_f is an ϵ-locally-Meir-Keeler type operator, having at least a fixed point.*

Proof. Let us consider $0 < \eta < \epsilon$ and $\delta > 0$ such that $A, B \in P_{cp}(X)$ and $\eta \leq H(A, B) < \eta + \delta$. We shall prove that $H(T_f(A), T_f(B)) < \eta$. For this purpose, let $u \in T_f(A)$ arbitrarily. Then there is $j \in \{1, \ldots, m\}$ and $x \in A$ such that $u = f_j(x)$. For $x \in A$, using again Lemma 2.1 we can choose $y \in B$ such that $d(x, y) \leq H(A, B) < \eta + \delta$.

If $d(x, y) \geq \eta$ then from the hypothesis we get $d(f_j(x), f_j(y)) < \eta$ and hence $D(u, T_f(B)) \leq d(f_j(x), f_j(y)) < \eta$.

If on the other hand $d(x, y) < \eta < \epsilon$ then $d(f_j(x), f_j(y)) < d(x, y)$ implies again that $D(u, T_f(B)) < \eta$.

As before we deduce that $H(T_f(A), T_f(B)) < \eta$ thus T_f is an ϵ-locally Meir-Keeler-type operator. The existence of the fixed point for T_f is now an easy application of Theorem 3.2. □

Remark 3.2. Matkowski and Wegrzyk considered in [23] the following more general condition:

vi) Matkowski-Wegrzyk type condition:

(6) for each $\eta > 0$ there is $\delta > 0$ such that $x, y \in X$, $\eta < d(x, y) < \eta + \delta$ we have $d(f(x), f(y)) \leq \eta$.

In [16] Jachymski proved that the condition (4) we have (3) $\wedge$ (6) but not conversely. It is an open question, if the fractal operator T_f satisfies (3) and (6) provided the operators f_i, $i \in \{1, \ldots, m\}$ have the properties (3) and (6).

Remark 3.3. Jachymski (see [16]), C. S. Wong (in [35]) and T. C. Lim (see[19]) proved that the Meir-Keeler type condition (4) is equivalent to other conditions of this type:

(4a) for any $\eta > 0$ there exists a $\delta > 0$ such that $x, y \in X$, $0 < d(x, y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$

(4b) for any $\eta > 0$ there exists a $\delta > 0$ such that $x, y \in X$, $0 \leq d(x,y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$

(4c) $\delta(\eta) > 0$, for each $\eta > 0$, where $\delta(\eta)$ denotes the modulus of uniform continuity of f.

(4d) there exists a lower semi-continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(0) = 0$, $\psi(\epsilon) > 0$, for every $\epsilon > 0$ and $\psi(d(f(x), f(y))) \leq d(x,y)$, for every $x, y \in X$

(4e) there exists a function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ with the properties $\lambda(0) = 0$, $\lambda(\epsilon) > 0$, for every $\epsilon > 0$ and for each $s > 0$ there exists $u > s$ with $\lambda(t) \leq s$, for each $t \in [s, u]$ such that $d(f(x), f(y)) \leq \lambda(d(x,y))$, for every $x, y \in X$, $x \neq y$.

Obviously, similar theorems for operators f_i, $i \in \{1, \ldots m\}$ satisfying the condition (4a)-(4e) can be proved.

Let us consider now the multi-valued case. In this respect, let $F_1, \ldots, F_m : X \to P_{cp}(X)$ be a finite family of upper semi-continuous multi-valued operators. We define the following multi-fractal operator(see [24 and [26]]) generated by $F = (F_1, \ldots, F_m)$:

$$T_F : P_{cp}(X) \to P_{cp}(X), \quad T_F(Y) = \cup_{i=1}^{m} F_i(Y). \tag{3.8}$$

By definition, $A^* \in P_{cp}(X)$ is a multi-self-similar set if $A^* = T_F(A^*)$.

Let us recall now some contractive-type conditions for multi-valued operators on a metric space (X, d)

i) The multi-valued operator $F : X \to P(X)$ is an a-contraction if and only if there exists $a \in [0, 1[$ such that:

$$H(T(x), T(y)) \leq ad(x,y), \text{ for each } x, y \in X.$$

ii) The multi-valued operator $F : X \to P(X)$ is a φ-contraction if and only if there exists the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H(T(x), T(y)) \leq \varphi(d(x,y)), \text{ for each } x, y \in X.$$

iii) The multi-valued operator $F : X \to P(X)$ is a Meir-Keeler type operator if and only if:

for each $\eta > 0$ there is $\delta > 0$ such that $\eta \leq d(x,y) < \eta + \delta$ implies $H(F(x), F(y)) < \eta$.

iv) The multi-valued operator $F : X \to P(X)$ is an ϵ-locally Meir-Keeler type operator (where $\epsilon > 0$) if and only if :

for each $\eta \in]0, \epsilon[$ there is $\delta > 0$ such that $\eta \leq d(x,y) < \eta + \delta$ implies $H(F(x), F(y)) < \eta$.

v) The multi-valued operator $F : X \to P(X)$ is contractive if and only if:

$$H(F(x), F(y)) < d(x,y), \text{ for each } x, y \in X, x \neq y.$$

An existence and uniqueness result for a multi-self-similar set is:

Theorem 3.5. *Let (X, d) be a complete metric space and $F_i : X \to P_{cp}(X)$, $i \in \{1, \ldots, m\}$ be a finite family of multi-valued Meir-Keeler type operators. Then the multi-fractal operator $T_F : P_{cp}(X) \to P_{cp}(X)$ defined by (3.8) is a (single-valued) Meir-Keeler type operator and $FixT_F = \{A^*\}$.*

Proof. Let us suppose that for each $\eta > 0$ there exists $\delta > 0$ such that $\eta \leq d(x,y) < \eta + \delta$ implies

$$H(F_i(x), F_i(y)) < \eta \text{ for } i \in \{1, \ldots, m\}. \tag{3.9}$$

From (2), it follows that F_i is contractive and hence F_i is upper semi-continuous, for $i \in \{1, \ldots, m\}$. As consequence $T_F : P_{cp}(X) \to P_{cp}(X)$.

Let us consider $\eta > 0$ and $Y_1, Y_2 \in P_{cp}(X)$ such that $\eta \leq H(Y_1, Y_2) < \eta + \delta$. We will prove that $H(T_F(Y_1), T_F(Y_2)) < \eta$.

For this purpose, let $u \in T_F(Y_1)$ be arbitrary. Then there exist $k \in \{1, \ldots, m\}$ and $y_1 \in Y_1$ such that $u \in F_k(Y_1)$. For this $y_1 \in Y_1$ there is $y_2 \in Y_2$ such that $d(y_1, y_2) \leq H(Y_1, Y_2) < \eta + \delta$.

If $d(y_1, y_2) \geq \eta$, then from (2) we get that $H(F_k(y_1), F_k(y_2)) < \eta$. It follows that there is $v \in F_k(y_2)$ such that $d(u, v) < \eta$ and hence $D(u, T_F(Y_2)) \leq d(u, v) < \eta$.

On the other hand if $0 < d(y_1, y_2) < \eta$ the from (2) we deduce that

$$H(F_k(y_1), F_k(y_2)) < d(y_1, y_2) < \eta$$

and as before $D(u, T_F(Y_2)) < \eta$.

Because $T_F(Y_1)$ is a compact set, we have that $\rho(T_F(Y_1), T_F(Y_2)) < \eta$. Interchanging the roles of $T_F(Y_1)$ and $T_F(Y_2)$ we obtain $\rho(T_F(Y_2), T_F(Y_1)) < \eta$ and the conclusion $H(T_F(Y_1), T_F(Y_2)) < \eta$ follows.

So $T_F : P_{cp}(X) \to P_{cp}(X)$ is a Meir-Keeler type operator and by Theorem 3.1. has a unique fixed point, i.e. $A^* \in P_{cp}(X)$ such that $T_F(A^*) = A^*$. □

The following abstract notion appear in [30] and [31].

Definition 3.1. Let (X, d) be a metric space and $f : X \to X$ an operator. By definition, f is a Picard operator if and only if for all $x \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by:

i) $x_0 = x$

ii) $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$

is convergent and its limit is the unique fixed of f.

Corollary 3.1. *Let (X, d) be a complete metric space and $F_i : X \to P_{cp}(X)$, $i \in \{1, \ldots, m\}$ be a finite family of multi-valued Meir-Keeler type operators. Then the multi-fractal operator T_F is a Picard operator*

Proof. The conclusion follows from Theorem 3.5. and Theorem 3.1. □

For the case of multi-valued operators satisfying to some locally contractive type conditions, we have the following results:

Theorem 3.6. *Let (X, d) be a complete ϵ-chainable metric space (where $\epsilon > 0$) and $F_i : X \to P_{cp}(X)$, $i \in \{1, \ldots, m\}$ be a finite family of multi-valued ϵ-locally Meir-Keeler type operators.*

Then the multi-fractal operator $T_F : P_{cp}(X) \to P_{cp}(X)$ given by (3.8.) is an (single-valued) ϵ-locally Meir-Keeler type operator, having a fixed point.

Proof. The proof runs exactly as in Theorem 3.5., but instead of using Theorem 3.1. , the conclusion follows from Theorem 3.2. □

Using an ϵ-locally Boyd-Wong type condition (see [7] and [34]) one can also prove:

Theorem 3.7. *Let (X, d) be a complete ϵ-chainable metric and let $F_i : X \to P_{cp}(X)$, $i \in \{1, \ldots, m\}$ be multi-valued operators such that*

$$H(F_i(x), F_i(y)) \leq k(d(x, y))d(x, y), \text{ for all } x, y \in X \tag{3.10}$$

with $0 < d(x, y) < \epsilon$, where $k : (0, \infty) \to (0, 1)$ is a real function with the property:

$$(P) \quad \begin{cases} \text{For each } 0 < t < \epsilon \text{ there exist } e(t) > 0 \text{ and } s(t) < 1 \\ \text{such that } k(r) \leq s(t) \text{ provided } t \leq r < t + e(t) \end{cases}$$

Then, the multi-fractal operator $T_F : P_{cp}(X) \to P_{cp}(X)$ given by (3.8.) satisfy the condition:

$$H(T_F(Y_1), T_F(Y_2)) \leq k(H(Y_1, Y_2))H(Y_1, Y_2),$$

for all $Y_1, Y_2 \in P_{cp}(X)$ with $0 < H(Y_1, Y_2) < \epsilon$ and has a fixed point.

Proof. Let $Y_1, Y_2 \in P_{cp}(X)$ such that $0 < H(Y_1, Y_2) < \epsilon$. Then

$$H(T_F(Y_1), T_F(Y_2)) \leq \max\{H(F_k(Y_1), F_k(Y_2))|\ k \in \{1, \ldots, m\}\} \leq$$
$$\leq k(H(Y_1, Y_2))H(Y_1, Y_2).$$

The conclusion follows now from Theorem 2 in H.K. Xu [34]. □

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