Seminar on Fixed Point Theory Cluj-Napoca, Volume 3, 2002, 323-326
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

ON SOME DIFFERENTIAL INEQUALITIES

DORIAN POPA AND NICOLAIE LUNGU

Technical University
Department of Mathematics
Str. C. Daicoviciu, 15
3400 Cluj-Napoca, Romania
e-mail: Popa.Dorian@math.utcluj.ro, nlungu@math.utcluj.ro

Abstract. Using a classical result on linear differential inequalities are established some results for Riccati inequality and the second order linear differential inequality.

1. Introduction

In [3], [4], [5] are given some operatorial inequalities and some applications to differential inequalities. All this results use the monotonicity of an operator and this fact implies various conditions on the coefficients of the considered differential inequality. E. Zeidler [6] gives also an operatorial inequality for a continuous, linear, positive operator with spectral radius less than one. The goal of this paper is to obtain some inequalities for the solutions of Riccati inequality and the second order linear differential inequality using a classical results for the first order differential inequality. This method implies few conditions on the coefficients of the inequality but the inequality has a locally character.

2. Main results

The following best known lemma will be used in the sequel.

Lemma 1. [1] *Let x_0, y_0 be real numbers $I = [x_0, +\infty)$ and $a, b \in C(I)$. Suppose that $y \in C^1(I)$ satisfies the following inequality*

$$y'(x) \leq a(x)y(x) + b(x), \quad x \geq x_0,\ y(x_0) = y_0. \tag{1}$$

Then

$$y(x) \leq y_0 \exp\left(\int_{x_0}^{x} a(t)dt\right) + \int_{x_0}^{x} b(s) \exp\left(\int_{s}^{x} a(t)dt\right) ds, \quad x \geq x_0. \tag{2}$$

If the converse inequality holds in (1), then the converse inequality holds in (2) too.

First we consider Riccati inequality.

Theorem 1. *Let x_0, y_0 be real numbers, $I = [x_0, +\infty)$ and $f, g, h \in C(I)$ be such that $f(x) \geq 0$ for all $x \geq x_0$ and the Cauchy problem*

$$y' = f(x)y^2 + g(x)y + h(x), \quad y(x_0) = \overline{y}_0, \quad y \in C^1(I) \tag{3}$$

has a unique solution denoted by $\overline{y}$ *in an interval* $I_1 = [x_0, x_1)$, $x_1 \in \overline{\mathbb{R}}$.

If $y \in C^1(I)$ *satisfies the following inequality*

$$y' \leq f(x)y^2 + g(x)y + h(x), \quad y(x_0) = y_0, \quad y_0 < \overline{y}_0, \tag{4}$$

then there exists an interval $I_2 = [x_0, x_2)$, $x_2 \in \overline{\mathbb{R}}$, *such that*

$$y(x) \leq \overline{y}(x) + \left[\frac{1}{y_0 - \overline{y}_0} exp\left(\int_{x_0}^{x} c(t)dt\right) - \int_{x_0}^{x} f(s)\exp\left(\int_{s}^{x} c(t)dt\right)ds\right]^{-1} \tag{5}$$

for all $x \in I_2$, *where* $c = -2f\overline{y} - g$.

If the converse inequality holds in (4) with $y_0 > \overline{y}_0$ *then the converse inequality holds in (5) too.*

Proof. Let y be a solution of the inequality (4). We have $y(x_0) - \overline{y}(x_0) < 0$, so there exists an interval $J' = [x_0, x_0')$, $x_0' \in \overline{\mathbb{R}}$, such that $y(x) - \overline{y}(x) < 0$ for all $x \in J'$. Putting

$$y = \overline{y} + \frac{1}{z}, \quad z \in C^1(J'), \quad z(x) < 0, \quad x \in J', \tag{6}$$

in the inequality (4) we get

$$z' + (f(x)\overline{y}(x) + g(x))z + f(x) \geq 0 \tag{7}$$

and using Lemma 1 we obtain

$$z(x) \geq z_0 \exp\left(\int_{x_0}^{x} c(t)dt\right) - \int_{x_0}^{x} f(s)\exp\left(\int_{s}^{x} c(t)dt\right)ds \tag{8}$$

where $c = -2f\overline{y} - g$ and $z_0 = z(x_0) = (y_0 - \overline{y}_0)^{-1}$.

The right hand of the relation (8) is negative because $f(x) > 0$ for $x \in I$ so there exists an interval $J'' = [x_0, x_0'')$, $x_0'' \in \overline{\mathbb{R}}$, where $z(x) < 0$ for $x \in J''$. Putting

$$I_2 = I_1 \cap J' \cap J'' \tag{9}$$

we obtain by (6) and (8)

$$y(x) = \overline{y}(x) + \left[\frac{1}{y_0 - \overline{y}_0} \exp\left(\int_{x_0}^{x} c(t)dt\right) - \int_{x_0}^{x} f(s)\exp\left(\int_{s}^{x} c(t)dt\right)ds\right]^{-1} \tag{10}$$

for all $x \in I_2$. The converse inequality results analogously.

Corollary 1. *If the conditions of Theorem 1 are satisfied then*

$$y(x) \leq \overline{y}(x), \quad x \in J_2 \tag{11}$$

or

$$y(x) \geq \overline{y}(x), \quad x \in J_2$$

if the converse inequality holds in (4) with $y_0 > \overline{y}_0$.

Proof. It follows from (10) taking account that the second term of the right hand is negative.

In the sequel we deal with the linear differential inequality of the second order.

Theorem 2. *Let* $I = [x_0, +\infty)$ *be a real interval and* $p, q \in C(I)$. *If* $y \in C^1(I)$ *satisfies the following inequality*

$$y'' + p(x)y' + q(x)y \leq 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_0', \quad y_0 \neq 0 \tag{12}$$

and $\overline{y}$ is the unique solution of the Cauchy problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{13}$$

then there exists an interval $I_1 = [x_0, x_1)$, $x_1 \in \overline{\mathbb{R}}$, such that

$$y(x) \le \overline{y}(x), \quad x \in I_1. \tag{14}$$

If the converse inequality holds in (12) then the converse inequality holds in (14).

Proof. By the continuity of $\overline{y}$ and $y_0 \neq 0$ it results that there exists an interval $I_1 = [x_0, x_1)$, $x_1 \in \overline{\mathbb{R}}$, such that $\overline{y}(x) \neq 0$ for all $x \in I_1$.

Suppose that $y_0 > 0$ and $\overline{y}(x) > 0$ for all $x \in I_1$. Let y be a solution of (12) and let z be defined by

$$z(x) = \frac{y(x)}{\overline{y}(x)}, \quad x \in I_1. \tag{15}$$

The inequality (12) becomes

$$\overline{y}(x)z'' + (2\overline{y}'(x) + p(x)\overline{y}(x))z' \le 0, \quad x \in I_1, \tag{16}$$

and denoting $z' = u$ we obtain

$$\overline{y}(x)u' + (2\overline{y}'(x) + p(x)\overline{y}(x))u \le 0, \quad x \in I_1. \tag{17}$$

Denoting

$$a(x) = -\frac{2\overline{y}'(x) + p(x)\overline{y}(x)}{\overline{y}(x)}, \quad x \in I_1 \tag{18}$$

we get

$$u'(x) \le a(x)u(x), \quad x \in I_1, \tag{19}$$

and taking account of Lemma 1 it results

$$u(x) \le u(x_0)\exp\left(\int_{x_0}^{x} a(t)dt\right), \quad x \in I_1. \tag{20}$$

But

$$u(x_0) = z'(x_0) = \frac{y'(x_0)\overline{y}(x_0) - y(x_0)\overline{y}'(x_0)}{(\overline{y}(x_0))^2} = 0$$

and in view of (20) we have $z'(x) \le 0$ for every $x \in I_1$. The function z is decreasing on I_1, hence

$$\frac{y(x)}{\overline{y}(x)} \le \frac{y(x_0)}{\overline{y}(x_0)} = 1, \quad x \in I_1 \tag{21}$$

which is equivalent with $y(x) \le \overline{y}(x)$ for every $x \in I_1$.

Now if $y_0 < 0$ we have $y(x) < 0$ for every $x \in I_1$ and the function z and u defined by (15) satisfy

$$u'(x) \ge a(x)u(x), \quad x \in I_1, \tag{22}$$

where a is given by (18). From Lemma 1 we get

$$u(x) \ge u(x_0)\exp\left(\int_{x_0}^{x} a(t)dt\right) = 0, \quad x \in I_1, \tag{23}$$

so the function z is increasing on I_1. It results

$$\frac{y(x)}{\overline{y}(x)} \geq \frac{y(x_0)}{\overline{y}(x_0)} = 1, \quad x \in I_1, \tag{24}$$

and since $\overline{y}$ is negative on I_1 we have

$$y(x) \leq \overline{y}(x), \quad x \in I_1. \tag{25}$$

References

[1] V. Lakshmikantham, S. Leela, A.A. Martynyuk, *Stability Analysis of Nonlinear Systems*, Marcel Dekker, Inc., New York and Basel, 1989.

[2] N. Lungu, *On some Gronwall-Bihari-type inequalities*, Libertas Mathematica, Vol.XX(2000), 67-70.

[3] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[4] J. Szarski, *Differential Inequalities*, PWN Warszawa, 1965.

[5] J. Scrhröder, *Operator Inequalities*, Academic Press, New York, 1980.

[6] E. Zeidler, *Fixed point theorems*, Springer-Verlag, Berlin, 1993.