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ON SOME FUNCTIONAL-INTEGRAL EQUATIONS WITH LINEAR MODIFICATION OF THE ARGUMENT

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Abstract. The purpose of this paper is to study the following functional-integral equation with linear modification of the argument:

$$x(t) = g(t, h(x)(t), x(t), x(0)) + \int_0^t K(t, s, x(\theta s))ds, \quad t \in [0, b], \ \theta \in [0, 1],$$

by the weakly Picard operators technique (see [4]-[6], [10]).
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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $h : C([0, b], X) \to C([0, b], X)$, $g \in C([0, b] \times X^3, X)$ and $K \in C([0, b] \times [0, b] \times X, X)$.

We consider the following functional-integral equation

$$x(t) = g(t, h(x)(t), x(t), x(0)) + \int_0^t K(t, s, x(\theta s))ds, \quad t \in [0, b], \ \theta \in [0, 1]. \tag{1.1}$$

By using the weakly Picard operators technique, we give a result about the solutions set and we study the data dependence of solutions set.

2. Some results about weakly Picard operators

Ioan A. Rus introduced the Picard operators class (PO) and the weakly Picard operators class (WPO) for the operators defined on a metric space and he gave basic notations, definitions and many results in this field in many papers ([3]-[7], [9], [10]).

In what follows we shall consider some of these results that are useful in our paper.

Let (X, d) be a metric space and $A : X \to X$ be an operator. We denote

$P(X) := \{Y \subset X |\ Y \neq \emptyset\}$;

$F_A := \{x \in X |\ A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \in P(X) |\ A(Y) \subset Y\}$;

$A^{n+1} := A \circ A^n, \ A^0 = 1_X, \ A^1 = A, \ n \in \mathbb{N}$.

Definition 2.1. (Rus [5]) The operator A is a Picard operator (PO) if there exists $x^* \in X$ such that:
(i) $F_A = \{x^*\}$;
(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.2. (Rus [4], [10]) The operator A is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

Definition 2.3. (Rus [4], [10]) If A is WPO then we consider the operator A^∞, $A^\infty : X \to X$, defined by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 2.1. $A^\infty(X) = F_A$.

Definition 2.4. (Rus [10]) If A is a WPO and $F_A = \{x^*\}$ then by definition the operator A is a PO.

Remark 2.2. If A is a PO then

$$F_{A^n} = F_A = \{x^*\}, \text{ for all } n \in \mathbb{N}^*.$$

Remark 2.3. If A is a WPO then

$$F_{A^n} = F_A \neq \emptyset, \text{ for all } n \in \mathbb{N}^*.$$

Remark 2.4. Some examples of PO and WPO and properties of PO and WPO have been given in the papers [3]-[10].

Definition 2.5. (Rus [10]) The operator A is a c-WPO if there exists $c > 0$ such that

$$d(x, A^\infty(x)) \leq c\, d(x, A(x)), \text{ for any } x \in X.$$

Example 2.1. If (X, d) is a complete metric space and the operator $A : X \to X$ is an a-contraction, then A is a c-WPO with $c = (1-a)^{-1}$.

Example 2.2. Let (X, d) be a complete metric space and $A : X \to X$. We suppose that there exists $a \in [0, 1[$ such that

$$d(A^2(x), A(x)) \leq a\, d(x, A(x)), \text{ for any } x \in X.$$

Then A is a c-WPO with $c = (1-a)^{-1}$.

We have

Theorem 2.1. (Rus [9]) *Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda$$

such that
(a) $X_\lambda \in I(A)$, $\lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is a Picard (c-Picard) operator for all $\lambda \in \Lambda$.

Remark 2.5. It is clear that
(i) $cardF_A = card\Lambda$;

(ii) if $\Lambda_1 \subset \Lambda$, then

$$card\left(F_A \cap \left(\bigcup_{\lambda\in\Lambda_1} X_\lambda\right)\right) = card\Lambda_1.$$

Theorem 2.2. (Rus [9]) *Let (X,d) be a metric space and $A_i : X \to X$, $i = 1,2$. We suppose that*

(i) the operator A_i is $c_i - WPO$, $i = 1,2$;

(ii) there exists $\eta > 0$ such that

$$d(A_1(x), A_2(x)) \leq \eta \quad \textit{for any } x \in X.$$

Then

$$H(A_1^\infty(X), A_2^\infty(X)) \leq \eta \max(c_1, c_2).$$

Here H stands for Hausdorff-Pompeiu functional.

Let $(X,d,\leq)$ be an ordered metric space and $A : X \to X$ an operator.

We have

Lemma 2.1. (Carl and Heikkilä [1]) *We suppose that*

(i) A is WPO;

(ii) A is monotone increasing.

Then the operator A^∞ is monotone increasing.

Lemma 2.2. (Abstract Gronwall Lemma, Rus [4], [6]) *We suppose that*

(i) A is PO and $F_A = \{x_A^*\}$;

(ii) A is monotone increasing.

Then

(a) $x \leq A(x)$ implies $x \leq x_A^*$.

(b) $x \geq A(x)$ implies $x \geq x_A^*$.

Lemma 2.3. (Rus [10]) *We suppose that*

(i) A is WPO;

(ii) A is monotone increasing;

(iii) $x, y \in X$ such that $x < y$, $x \leq A(x)$ and $y \geq A(y)$.

Then

(a) $x \leq A^\infty(x) \leq A^\infty(y) \leq y$;

(b) $A^\infty(x)$ is the minimal fixed point of A in $[x,y]$ and $A^\infty(y)$ is the maximal fixed point of A in $[x,y]$.

Let $(X,d,\leq)$ be an ordered metric space and $A : X \to X$, $B : X \to X$, $C : X \to X$ three operators.

We have

Lemma 2.4. (Carl and Heikkilä [1]) *We suppose that*

(i) $A \leq B \leq C$;

(ii) A, B, C are WPO;

(iii) B is monotone increasing.

Then $x \leq y \leq z$ implies

$$A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

Remark 2.6. If A, B, C are as in the Lemma 2.4 and B is PO and $F_B = \{x_B^*\}$ then

$$A^\infty(x) \le x_B^* \le C^\infty(x) \text{ for any } x \in X.$$

3. Functional-differential equation with linear modification of the argument

Let $(X, \|\cdot\|)$ a Banach space and the space $C([0,b], X)$ endowed with the Bielecki norm $\|\cdot\|_\tau$, defined by

$$\|x\|_\tau := \max_{t\in[0,b]} \|x(t)\| e^{-\tau t}, \quad \tau > 0.$$

We consider the functional-integral equation (1.1) and we suppose that the following conditions are satisfies

(c_1) there exists $l > 0$ such that

$$\|h(x)(t) - h(y)(t)\| \le l\|x(t) - y(t)\|$$

for all $x, y \in C([0,b], X)$ and all $t \in [0,b]$;

(c_2) there exist $l_1 > 0$, $l_2 > 0$ such that

$$\|g(t, u_1, v_1, w) - g(t, u_2, v_2, w)\| \le l_1\|u_1 - u_2\| + l_2\|v_1 - v_2\|,$$

for all $t \in [0,b]$, $u_i, v_i, w \in X$, $i = 1, 2$;

(c_3) there exists $l_3 > 0$ such that

$$\|K(t, s, u) - K(t, s, v)\| \le l_3\|u - v\|,$$

for all $t, s \in [0,b]$ and $u, v \in X$;

(c_4) $l_1 l + l_2 < 1$;

(c_5) $g(0, h(x)(0), x(0), x(0)) = x(0)$ for any $x \in C([0,b], X)$.

We have

Theorem 3.1. *We suppose that the conditions $(c_1) - (c_5)$ are satisfied. If $S \subset C(I, X)$, $I \subseteq [0,b]$ is the solution set of the equation (1.1) then $cardS = cardX$.*

Proof. Let $A : C([0,b], X) \to C([0,b], X)$ be defined by

$$A(x)(t) := g(t, h(x)(t), x(t), x(0)) + \int_0^t K(t, s, x(\theta s))ds. \tag{3.1}$$

Let $\lambda \in X$ and

$$X_\lambda := \{x \in C([0,b], X) |\ x(0) = \lambda\}.$$

Then

$$C([0,b], X) = \bigcup_{\lambda\in X} X_\lambda$$

is a partition of $C([0,b], X)$. From (c_5) we have that $X_\lambda \in I(A)$. Let

$$A_\lambda := A|_{X_\lambda} : X_\lambda \to X_\lambda.$$

From $(c_1) - (c_3)$ it follows that

$$\|A_\lambda(x) - A_\lambda(y)\|_\tau \le \left(l_1 l + l_2 + \frac{l_3}{\theta\tau}\right)\|x - y\|_\tau,$$

for all $x, y \in C([0,b], X)$, $\lambda \in X$, $\tau > 0$.

Because of the condition (c_4) we can choose τ large enough such that $l_1 l + l_2 + \dfrac{l_3}{\theta\tau} < 1$. Then $A_\lambda : (X_\lambda, \|\cdot\|_\tau) \to (X_\lambda, \|\cdot\|\tau)$ is a contraction, i.e., A_λ is PO for all $\lambda \in X$. Moreover A_λ is c-PO with the constant

$$c = \left(1 - l_1 l - l_2 - \frac{l_3}{\theta\tau}\right)^{-1}.$$

From the Theorem 2.1 we have that the operator A is c-WPO. So we have that $cardS = cardX$.

Theorem 3.2. *We consider the equation (1.1) under the following conditions:*
(i) the conditions $(c_1) - (c_5)$*;*
(ii) the operators $h(\cdot), g(t,\cdot,\cdot,\cdot), K(t,s,\cdot)$ *are monotone increasing.*
Let x *and* y *be two solutions of the equation (1.1).*
If $x(0) \leq y(0)$*, then* $x(t) \leq y(t)$ *for all* $t \in [0,b]$*.*

Proof. Let X_λ be as in the proof of the Theorem 3.1. Then $x \in X_{x(0)}$ and $y \in X_{y(0)}$. Moreover $x = A^\infty(x_1)$ for any $x_1 \in X_{x(0)}$ and $y = A^\infty(y_1)$ for any $y_1 \in X_{y(0)}$. If $u \in X$ then we denote by $\widetilde{u}$ the operator $\widetilde{u} \in C([0,b], X)$ defined by $\widetilde{u}(t) = u$, $t \in [0,b]$. We have that

$$\widetilde{x(0)} \in X_{x(0)}, \quad \widetilde{y(0)} \in X_{y(0)} \text{ and } \widetilde{x(0)} \leq \widetilde{y(0)}.$$

Because of the conditions of this theorem, the operator A given by the relationship (3.1), satisfies the conditions from Lemma 2.1. So, the operator A^∞ is monotone increasing. It follows that $A^\infty(\widetilde{x}(0)) \leq A^\infty(\widetilde{y}(0))$, i.e., $x \leq y$.

We consider the equations

$$x(t) = g_1(t, h(x)(t), x(t), x(0)) + \int_0^t K_1(t,s,x(\theta s))ds, \tag{3.2}$$

$$x(t) = g_2(t, h(x)(t), x(t), x(0)) + \int_0^t K_2(t,s,x(\theta s))ds, \tag{3.3}$$

$$x(t) = g_3(t, h(x)(t), x(t), x(0)) + \int_0^t K_3(t,s,x(\theta s))ds, \tag{3.4}$$

where $t \in [0,b]$ and $\theta \in [0,1]$, for which the same conditions $(c_1) - (c_5)$ are satisfied.

Let S_1 be the solutions set of the equation (3.2) and S_2 be the solutions set of the equation (3.3).

We have

Theorem 3.3. (data dependence theorem) *Suppose that there exist* $\eta_1 > 0$*,* $\eta_2 > 0$ *such that*

$$\|g_1(t,u,v,w) - g_2(t,u,v,w)\| \leq \eta_1 \text{ for all } t \in [0,b],\ u,v,w \in X$$

and

$$\|K_1(t,s,u) - K_2(t,s,u)\| \leq \eta_2 \text{ for all } t,s \in [0,b],\ u \in X.$$

Then

$$H_\tau(S_1,S_2)\le \frac{\eta_1+b\eta_2}{1-l_1l-l_2-\dfrac{l_3}{\theta\tau}}.$$

Proof. The conditions of this theorem imply those of Theorem 2.2 (see the proof of the Theorem 3.1).

Remark 3.1. Let $\alpha,\beta\in\mathbb{R}$ be, where $\alpha<\beta$. Consider

$$Y:=\{x\in C([0,b],X)|\ \alpha\le x(0)\le\beta\}.$$

Then we have that

$$H_\tau(S_1\cap Y,S_2\cap Y)\le \frac{\eta_1+b\eta_2}{1-l_1l-l_2-\dfrac{l_3}{\theta\tau}}.$$

Theorem 3.4. *We consider the equations (3.2), (3.3) and (3.4) with the conditions* $(c_1)-(c_5)$ *given before. We suppose that* $g_1\le g_2\le g_3$, $K_1\le K_2\le K_3$ *and that the operators* $h(\cdot)$, $g_2(t,\cdot,\cdot,\cdot)$, $K_2(t,s,\cdot)$ *are monotone increasing. Let* v_1,v_2,v_3 *be the corresponding solutions of the equations (3.2), (3.3) and respectively (3.4). If* $v_1(0)\le v_2(0)\le v_3(0)$ *then* $v_1\le v_2\le v_3$.

Proof. Let $A_i:C([0,b],X)\to C([0,b],X)$, $i=1,2,3$, given by

$$A_i(x)(t):=g_i(t,h(x)(t),x(t),x(0))+\int_0^t K_i(t,s,x(\theta s))ds,$$

$t\in[0,b]$, $\theta\in[0,1]$, $i=1,2,3$.

The operators A_i, $i=1,2,3$ are WPO, the operator A_2 is monotone increasing and $A_1\le A_2\le A_3$. So we are in the conditions of Lemma 2.4. It follows that $A_1^\infty\le A_2^\infty\le A_3^\infty$. We have

$$\widetilde{v_1(0)}\le\widetilde{v_2(0)}\le\widetilde{v_3(0)},\quad v_1\in X_{v_1(0)},\quad v_2\in X_{v_2(0)},\quad v_3\in X_{v_3(0)}.$$

Therefore

$$v_1=A_1^\infty(\widetilde{v_1(0)})\le A_2^\infty(\widetilde{v_2(0)})=v_2\le A_3^\infty(\widetilde{v_3(0)})=v_3,$$

i.e., $v_1\le v_2\le v_3$.

Remark 3.2. The equation obtained from (1.1) when $\theta=1$ have been studied by Rus in the paper [9].

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