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CONTROLLABILITY OF CERTAIN DIFFERENTIAL EQUATIONS AND INCLUSIONS

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Abstract. Some well known criteria of controllability and observability of linear and time invariant systems in $\mathbb{R}^n$ has been extended in various directions. First we review briefly this topic, then we introduce two necessary and sufficient criteria of approximately controllability for differential equations as well as a sufficient condition of approximate null controllability to quasi-linear differential inclusions. For the last result the method of investigation is based on a kind of Filippov existence theorem.

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1. Introduction

This section is devoted to introduce the ideas of controllability and observability.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Denote by W an open neighborhood of $x_0 \in \mathbb{R}^n$. Consider the following *control system*

$$(\mathrm{S}_1) \qquad \begin{cases} x'(t) &= f(t, x(t), u(t)) \\ y(t) &= h(x(t)), \qquad x(t_0) = x_0,\ t \in T \end{cases}$$

where T is an interval (bounded or not), $t_0 \in \mathrm{Int}(T)$, $T \ni t \mapsto x(t) \in \mathbb{R}^n$ is the *state* trajectory, $T \ni t \mapsto u(t) \in U \subset \mathbb{R}^m$ is the *control* function, and $T \ni t \mapsto y(t) \in \mathbb{R}^p$ is the *output* curve.

Given system (S_1) initialized at x_0, the map

$$\mathcal{S}_{x_0} : \{T \ni \mapsto u(t)\} \ \rightarrow \ \{T \ni \mapsto y(t)\}$$

is called the *input-output* map.

Example. If f and h are linear functions and the dynamics of system (S_1) is time invariant, then we get the simplest case

$$(\mathrm{S}_l) \qquad \begin{aligned} x'(t) &= Ax(t) + Bu(t), \qquad A \in M_{n\times n},\ B \in M_{n\times m}, \\ y(t) &= Cx(t), \qquad C \in M_{p\times n}. \end{aligned}$$

x_0 x_1

Suppose we are given a system (S_1) and an initial state x_0. Let x_1 be another state. Is it possible to choose a control u to steer (S_1) from x_0 to x_1? This is often referred to as *reachability*, here x_1 is reachable from x_0.

If so, x_1 is *accessible* from x_0. What are the accessible states? Is x_1 accessible from x_0 locally?

Roughly speaking, (S_1) is *controllable* if every state is accessible from every other state. What criteria (algebraic, geometrical) tell us when (S_1) is controllable, [21]?

This time we are given an output "record" $t \mapsto y(t)$, $t \in [t_0, t_f]$. We ask what information about the states can be obtained from such a record. Two initial states $x_0, {x_0}'$ are *indistinguishable* if no matter what control we use, the corresponding trajectories always produce the same output record.

${x_0}'$ ${x_1}'$ $y(T)$

h

h

x_0 x_1 $y(0)$

What are the indistinguishable/distinguishable states? Can states be locally distinguished?

(S_1) is *observable* if any state is distinguishable from any other state. What criteria is available here?

We will supply some answers to the above questions.

In a sense, observability is a "dual" notion to controllability, as it will result from section 2 and section 4.

The present lecture covers the following topics:

- controllability and observability in the time invariant case in finite dimensional spaces, [22], [39];

- controllability in the non-linear case in finite dimensional spaces, fixed point method, [33], [11];
- controllability and observability of convex processes in finite dimensional spaces, [3], [19], [4], [5], [8];
- constraint controllability in infinite dimensional Banach spaces [30], [1], [27]
- approximate null controllability of certain differential inclusions in infinite dimensional Banach spaces.

At the same time many important topics in this field remain uncovered by the present paper

- genericity, [10];
- delay systems, [18];
- stabilization of certain continuous or discrete systems, [39].

Some methods of approach are used regarding these problems

- algebraic, [18];
- geometrical, [10];
- analytical, [29];
- non-smooth, [3], [4], [5], [26].

2. Linear case in finite dimensional spaces

In this case we have system (S_l), i.e.,

$$x'(t) = Ax(t) + Bu(t), \qquad A \in M_{n\times n},\ B \in M_{n\times m},$$
$$y(t) = Cx(t), \qquad C \in M_{p\times n}.$$

If the control function u is (at least) Lebesgue integrable, the general solution of the above system is

$$x(t) = e^{At}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau, \quad t \in T, \tag{1}$$

so that the output is given by

$$y(t) = Ce^{At}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau. \tag{2}$$

Following [36] we say that the system (S_1) is *(completely) state*

(i) *approximately controllable* on the finite interval $[t_0, t_f] \subset T$ if given $\varepsilon > 0$ and two arbitrary initial and final points x_0 and x_f in the state space there is an admissible controller $u(\cdot)$ on $[t_0, t_f]$ steering x_0, along a solution curve of (S_1), to an ε-ball of x_1, that is such that $\|x(t_f, t_0, x_0, u) - x_1\| \le \varepsilon$.

(ii) *exactly controllable* on $[t_0, t_f]$ if $\varepsilon = 0$ in (i).

To system (S_1) let us introduce the so-called *controllability Gramian*

$$G(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_f-\tau)}BB^Te^{A^T(t_f-\tau)}d\tau, \tag{3}$$

and the *controllability matrix*

$$Q = [B, AB, A^2B, \cdots, A^{n-1}B]. \tag{4}$$

Characterization theorem

Theorem 2.1.. *For the linear time invariant system* (S_l) *the following statements are equivalent*

(a) (S_l) *is completely controllable;*
(b) *the controllability Gramian satisfies* $G(t_0, t) > 0$ *for all* $t > t_0$;
(c) *the controllability matrix* Q *has rank* n *(Kalman criterion);*
(d) *the rows of* $e^{At}B$ *are linearly independent functions of time;*
(e) *the rows of* $(sI - A)^{-1}B$ *are linearly independent functions of* s;
(f) *rank*$([A - \lambda I, B]) = n$, *for all* λ *(suffices to check only the eigenvalues of* A*);*
(g) $v^T B = 0$ *and* $v^T A = \lambda v^T \implies v = 0$ *(Popov-Belevich-Hautus test);*
(h) *given any set* Γ *of numbers in* $\mathbb{C}$ *there exists a matrix* K *such that the spectrum of* $A + BK$ *is equal to* Γ *(pole placement condition).*

The non-autonomous system (S_l) is *completely observable* on the finite interval $[t_0, t_f]$ if for each admissible control on $[t_0, t_f]$ and for every two responses $x(t)$ and $\overline{x}(t)$ with distinct initial conditions, the outputs $Cx(t)$ and $C\overline{x}(t)$ are distinct.

Given the system $[A, B, C]$, the *adjoint system* is defined as $[-A^T, C^T, B^T]$.

By duality we can state

Theorem 2.2.. *For the linear time invariant system* (S_l) *we have that the system* $[A, B, C]$ *is completely controllable if and only if the system* $[-A^T, C^T, B^T]$ *is completely observable.*

Remark 2.1. *Based on theorem 2.2 all statements of theorem 2.1 can be converted in statements on observability.* $\triangle$

3. Controllability to some non-linear cases, fixed point method

Many times the heavy duty in establishing a controllability result is carried out by a fixed point theorem: the Leray-Schauder fixed point theorem in [14] and [9], the Banach and the Karlin-Bohnenblust fixed point theorems in [11]. In [12] this job is carried out by an inverse function theorem. For fixed point theorems see, e.g., [35], [33], [32], and [34] as well as the references therein.

As a quite general rule the case considered here concerns with problems having the dynamics governed by ([11])

$$x' = Ax + Nx, \quad x(t_0) = x_0, \tag{5}$$

where A is the linear part and N the non-linear part.

In [14] it is studied the following finite-dimensional control system

$$x'(t) = A(t, x(t))x(t) + B(t, x(t))u(t) \quad x(t_0) = x_0, \tag{6}$$

where the elements of matrices A and B are continuous functions of x for fixed t, piecewise continuous functions of t for fixed x, and are bounded over a finite time-interval.

In order to formulate the problem in terms of a fixed-point theorem, it is assumed that the linear system

$$x'(t) = A(t, z(t))x(t) + B(t, z(t))u(t) \tag{7}$$

is controllable, where z is a specified function belonging to the space $C[t_0, t_f; \mathbb{R}^n]$. The solution in terms of the state transition matrix $S(t, t_0; z)$ is

$$x(t) = S(t, t_0; z)x_0 + \int_{t_0}^{t} S(t, \tau; z)B(\tau, z)u(\tau)d\tau. \tag{8}$$

Define

$$H(t_0, \tau; z) = S(t_0, \tau; z)B(\tau, z),$$
$$G(t_0, t; z) = \int_{t_0}^{t} H(t_0, \tau; z)H^T(t_0, \tau; z)d\tau$$

Then the particular control function

$$u(t) = H^T(t_0, t; z)G(t_0, t_f; z)^{-1}[S(t_f, t_0; z)^{-1}x_1 - x_0] \tag{9}$$

drives the system (7) from x_0 to x_1 in a finite time. Substituting $u(t)$ into the right-hand side of (8) yields the nonlinear operator

$$(Pz)(t) = S(t, t_0; z)x_0 + \int_{t_0}^{t} S(t, \tau; z)B(\tau, z)H^T G^{-1}[S(t_f, t_0; z)^{-1}x_1 - x_0]d\tau. \tag{10}$$

Clearly $Pz(t_0) = x_0$ and $Pz(t_f) = x_1$. Thus if a fixed point of the operator P is obtained, control (9) steers system (6) from x_0 to x_1 in time $t_f - t_0$.

Let us introduce a preliminary result based on the Schauder fixed point theorem, [33].

Lemma 3.1.. ([14]) *Suppose that*

(i) *the elements of matrices A and B are continuous functions of x for fixed t, piecewise continuous functions of t for fixed x, and are bounded over a finite time-interval;*

(ii) *there exists a constant $d > 0$ such that*

$$\inf_{z \in C([t_0, t_f])} \det G(t_f, z) \geq d$$

for a $t_f > t_0$.

Then for any $x_0, x_1 \in \mathbb{R}^n$ the operator P has a fixed point in $C([t_0, t_f]; \mathbb{R}^n)$.

Now the controllability of system (6) follows as

Theorem 3.1.. ([14]) *The system (6) is globally completely controllable at the instant t_0 if there are fulfilled the assumptions (i) and (ii) of lemma 3.1 for a $t_f > t_0$.*

Remarks 3.1. (a) *In a similar way sufficient conditions for controllability of the perturbed quasi-linear system*

$$x' = A(t, x, u)x + B(t, x, u)u + f(t, x, u). \tag{11}$$

are studied, [13].

(b) *Observability problems can be studied also by fixed point methods,* [11]. △

4. Controllability and observability of convex processes in finite dimensional spaces

Differential inclusions are natural extensions of differential equations, [2], [15].

A *convex process* A from $\mathbb{R}^n$ to itself is a set-valued map satisfying

$$\forall\, x, y \in \operatorname{Dom} A, \quad \lambda,\, \mu \geq 0, \quad \lambda A(x) + \mu A(y) \subset A(\lambda x + \mu y), \tag{12}$$

that is, a set-valued map whose graph is a convex cone.

We associate with a strict closed convex process A the Cauchy problem for the differential inclusion: find an absolutely continuous function $x(\cdot)$ satisfying

$$x'(t) \in A(x(t)), \quad \text{a.e., } t \in [t_0, t_f] \quad x(t_0) = 0. \tag{13}$$

We denote the *reachable set at time* t by

$$R_t := \{x(t) \mid x \text{ is a solution of (13)}\} \tag{14}$$

and by

$$R := \cup_{t > t_0} R_t \tag{15}$$

the *reachable set*. We say that the differential inclusion (13) (or the convex process A) is *controllable* if the reachable set R is equal to $\mathbb{R}^n$.

It is well-known that for linear problems the reachable sets are invariant. It is useful to extend the concept of invariant subspace by a linear operator. This can be done in two ways: let A a convex process and P be a closed convex cone contained in Dom$A$. We recall that the *tangent cone* $T(P; x)$ at a point $x \in P$ is defined by

$$T(P; x) := \operatorname{cl}\left(\cup_{h>0} \frac{1}{h}(P - x)\right). \tag{16}$$

We say that P is *invariant by* A if

$$\forall\, x \in P, \quad A(x) \subset T(P; x) \tag{17}$$

and that P is a *viability domain* for A if

$$\forall\, x \in P, \quad A(x) \cap T(P; x) \neq \emptyset. \tag{18}$$

A real number λ such that $\operatorname{Im}(A - \lambda I) \neq \mathbb{R}^n$ is an *eigenvalue of* A.

Let A a convex process. Its *transpose* A^* is defined by

$$p \in A^*(q) \Longleftrightarrow \forall\, (x, y) \in \mathcal{G}(A), \quad \langle p, x\rangle \leq \langle q, y\rangle. \tag{19}$$

We associate with the differential inclusion (13) the *adjoint inclusion*

$$-q'(t) \in A^*(q(t)), \quad \text{a.e. } t \in [t_0, t_f]. \tag{20}$$

We introduce the cones Q_{t_f} and Q by

$$\begin{aligned} &\text{(i)} \quad Q_{t_f} := \{\eta \mid \exists q(\cdot) \text{ solution to (20) satisfying } q(t_f) = \eta\}, \\ &\text{(ii)} \quad Q := \cap_{t > t_f} Q_t. \end{aligned} \tag{21}$$

To say that $Q = 0$ amounts to saying that the only solution to (20) defined on $[0, \infty[$ is $q = 0$ or that the adjoint system is *observable*.

If there is a solution $q \neq 0$ and $\lambda \in \mathbb{R}$ to the inclusion $\lambda q \in A^*(q)$, the q is an *eigenvector* of A^*. Then there hold the next result

Theorem 4.1.. ([3]) *Let A be a strict convex process. The following conditions are equivalent.*
(a) *differential inclusion* (13) *is controllable (i.e., $R = \mathbb{R}^n$);*
(b) *differential inclusion* (13) *is controllable at some time $t > 0$ (i.e. $R_t = \mathbb{R}^n$);*
(c) *the adjoint system* (20) *is observable (i.e., $Q = \{0\}$);*
(d) *the adjoint system* (20) *is observable at some time t (i.e., $Q_t = \{0\}$);*
(e) $\mathbb{R}^n$ *is the smallest closed convex cone invariant by A*;
(f) $\{0\}$ *is the largest closed convex cone which is a viability domain for A^**;
(g) *A has neither proper invariant subspaces nor eigenvalues;*
(h) *A^* has neither proper invariant subspaces nor eigenvectors.*

By duality we infer the equivalent characteristic properties of system (20)

Theorem 4.2.. ([3]) *Let A be a strict convex process. The following conditions are equivalent*
(a) *the adjoint inclusion* (20) *is observable;*
(b) *the adjoint inclusion* (20) *is observable at time $t > t_0$ for some t*;
(c) $\{0\}$ *is the largest closed convex cone which is a viability domain for A^**;
(d) *A^* has neither proper invariant subspaces nor eigenvectors.*

Other results on the same vein may be found in [3], [19], [4], and [5].

5. Constraint controllability in infinite dimensional Banach spaces

Much attention has been paid to extend in various direction the well-known Kalman criterion on the controllability of autonomous linear processes in $\mathbb{R}^n$, [22, p. 81]. The present section extends to the case of Banach spaces two results given earlier in [27] and [30] for the case of Hilbert spaces. Paper [27] exhibits a necessary and sufficient result on controllability, sharpening a previous one introduced in [30].

Let us introduce some notations. If X is a topological space and $Y \subset X$, then by $\operatorname{int} Y$ and $\operatorname{cl} Y$ we denote the set of interior points, and the closure of Y, respectively. If X is a Banach space, then by $\mathcal{L}(X)$ we denote the space of linear and bounded operators from X in X. X^* is the Banach space of the linear and continuous functionals on X. Let F be a multifunction from a σ-algebra to a Banach space. By S_F we denote the set of measurable selections from F, while by S_F^1 we denote the set of Bochner integrable selections from F, [24], [25], [26].

5.1. The first result.

Let $T := [t_0, t_f]$, μ the Lebesgue measure on T, X and Y be separable real Banach spaces. Consider

(U) a measurable multifunction $U : T \rightsquigarrow Y$ having nonempty and closed values such that $S_U^1 \neq \emptyset$;

(B) a mapping $B : T \times Y \to X$ measurable in the first variable and continuous in the second one and fulfilling an estimation of the following form

$$\|B(t,u)\| \leq l(t) + b\|u\|, \text{ a.e., where } l \in L_+^1,\ b \geq 0;$$

(A) a family $\{A(t)\}_{t\in T}$ of linear and densely defined operators generating an evolution operator $S : \Delta = \{(t,s) \in T\times T \mid t_0 \le s \le t \le t_f\} \to \mathcal{L}(X)$, i.e.

$S(t,t) = I$, $\forall t \in T$, I is the identity,

$S(t,\tau)S(\tau,s) = S(t,s)$, $\forall\ t_0 \le s \le \tau \le t_f$,

$S : \Delta \to \mathcal{L}(X)$ is continuous in the strong operator topology, [31].

Under the above conditions our attention focuses on the following system

$$x'(t) = A(t)x(t) + B(t,u(t)), \quad t \in T,\ u \in S_U. \tag{22}$$

We will see some properties of the mild solutions of the system (22), i.e. given $x_0 \in X$ (as initial value) a mild solution of (22) is a continuous function $x \in C(T,X)$ which can be written as

$$x(t) = S(t,t_0)x_{t_0} + \int_{t_0}^{t} S(t,s)B(s,u(s))ds,\ t \in T, \tag{23}$$

where u is a measurable selection of the multifunction U such that $B(\cdot,u(\cdot)) \in L^1$.

The reachable set from x_0 at time $t \in T$ is defined as

$$R(t,x_0) = \{x(t) \in X \mid x(\cdot) \text{ is a mild solution of (22)}\}. \tag{24}$$

From (23) and (24) easily follows that

$$R(t,y_0) = S(t,t_0)(y_0 - x_0) + R(t,x_0).$$

The latter equality means that the topological properties of the reachable set are invariant under translations.

Different notions of controllability are investigated in [36] and [37]. We now recall here only one. The system (22) is said to be *approximately controllable on* T if for every $x_0 \in X$ we have that $\operatorname{int}\operatorname{cl}R(t_f,x_0) \neq \emptyset$. If $B(t,u) = B(t)u$, then this definition agrees with the corresponding one given in [36]. We denote $S(t,s)B(s,U(s)) = \{S(t,s)B(s,u) \mid u \in U(s),\ t_0 \le s \le t \le t_f\}$.

Starting from [30] and [27] we state our

Theorem 5.1.. *Admit the* (U), (B) *and* (A) *hypotheses and consider system* (22). *Moreover, suppose that*

(i) $\mu\{t \in T \mid S(t_f,t)B(t,U(t))$ *is not a singleton*$\} > 0$,

(ii) *the multifunction* $T \ni t \mapsto S(t_f,t)B(t,U(t))$ *is graph measurable.*

Then system (22) *is approximate controllable on* T *if and only if there exists no* $x^* \in X^* \setminus \{0\}$ *so that* $x^*(S(t_f,t)B(t,U(t))) = constant$, *a.e. on* T.

Proof. *Necessity.* Suppose that there exists $x^* \in X^* \setminus \{0\}$ with $x^*(S(t_f,t)B(t,U(t))) = constant$, a.e. on T. Then there exists $u \in S^1_U$ so that if $c(t) := x^*(S(t_f,t)B(t,u(t)))$, it follows $c(\cdot) \in L^1$ and $R(t_f,0) \neq \emptyset$. Let $x \in R(t_f,0)$. Then there exists $u \in S^1_U$ such that

$$x(t_f) = \int_0^{t_f} S(t_f,t)B(t,u(t))\,dt.$$

Taking into account [38, Corollar V.5.2, p.134] it follows that

$$\begin{aligned} x^*(x) &= x^*\left(\int_{t_0}^{t_f} S(t_f,t)B(t,u(t))dt\right) = \int_{t_0}^{t_f} x^*(S(t_f,t)B(t,u(t))dt \\ &= \int_{t_0}^{t_f} x^*(c(t))dt = k \in \mathbb{R}. \end{aligned}$$

Let $z \in V := \{z \in X \mid x^*(z) = k\}$. V is a closed hyperplane and $\mathrm{cl}(R(a,0)) \subset V$. Hence $\mathrm{int\,cl}(R(t_f,0)) \subset \mathrm{int}(V)$, so $\mathrm{int\,cl}(R(t_f,0)) = \emptyset$, i.e. our (22) system is not controllable.

Sufficiency. The idea is simple: to choose two integrable selections from $S(b,\cdot)B(\cdot, U(\cdot))$ far away one from the other such that the corresponding solutions of system (22) to be also sufficiently far one from the other.

From the Castaing representation theorem, [20, theorem 5.6] or [24, theorem 4.2.3], it follows that there exists $\{u_n\}_{n\geq 1}$ a countable family of measurable functions such that $U(t) = \mathrm{cl}\{u_n(t) \mid n \geq 1\}$, for all $t \in T$.

Let us choose an arbitrary, but fixed $x^* \in X^* \setminus \{0\}$. Then for t in a subset of T having a strictly positive measure there exist $v_1, v_2 \in S(t_f,t)B(t,U(t))$ with $x^*(v_1 - v_2) \neq 0$.

For a while we admit that U has bounded values, too. Later on we will remove this extra assumption. Define the following mappings

$$\begin{aligned} M(t) &= \sup\{x^*(S(t_f,t)B(t,U(t)))\} = \sup_n\{x^*(S(t_f,t)B(t,u_n(t)))\}, \\ m(t) &= \inf\{x^*(S(t_f,t)B(t,U(t)))\} = \inf_n\{x^*(S(t_f,t)B(t,u_n(t)))\}. \end{aligned}$$

From the hypotheses it follows that

$$\|x^*(S(t_f,t)B(t,U(t)))\| \leq \|x^*\| \cdot \|S(t_f,t)\|(l(t) + b\|u(t)\|),$$

and from the boundedness of U we may write

$$-\infty < m(t) \leq M(t) < +\infty, \qquad \text{a.e. on } T.$$

Also we have that the mappings m and M are measurable on T and, at the same time,

$$\eta(t) := [M(t) - m(t)]/2, \quad t \in T,$$

is measurable. From (i) it follows that if $C := \{t \in T \mid \eta(t) > 0\}$, then $\mu(C) > 0$. Define $\varepsilon : C \to \mathbb{R}_+$ as $\varepsilon(t) := \eta(t)/2$, $t \in C$. Since the differences $M(t) - \varepsilon(t)$, respectively $m(t) - \varepsilon(t)$ are well defined for all $t \in C$ we may consider the multifunctions $L_i(h) : C \rightsquigarrow Y$, $i = 1,2$ defined by

$$\begin{cases} L_1(h)(t) := \{u \in U(t) \mid x^*(S(t_f,t)B(t,u)) \geq M(t) - \varepsilon(t)\}, \\ L_2(h)(t) := \{u \in U(t) \mid x^*(S(t_f,t)B(t,u)) \leq m(t) + \varepsilon(t)\}. \end{cases} \tag{25}$$

Let us check that L_1 and L_2 are graph measurable. Note that

$$graph\, L_1 = graph\, U \cap graph\, F_1, \quad graph\, L_2 = graph\, U \cap graph\, F_2,$$

where

$$C \ni t \mapsto F_1(t) := \{x \in Y \mid f_1(t,x) \geq 0\},$$
$$C \ni t \mapsto F_2(t) := \{x \in Y \mid f_2(t,x) \leq 0\},$$

and

$$\begin{cases} f_1(t,x) := x^*(S(a,t)B(t,x)) - (M(t) - \varepsilon(t)), & t \in C,\ x \in Y, \\ f_2(t,x) := x^*(S(a,t)B(t,x)) - (m(t) + \varepsilon(t)), & t \in C,\ x \in Y. \end{cases} \tag{26}$$

Invoking [20, theorem 6.4], we infer the measurability of F_1 and F_2. Hence L_1 and L_2 are graph measurable. By the Aumann selection theorem, [20, theorem 5.2], we can choose two measurable functions u_1 and u_2 such that

$$u_i : C \to Y,\ u_i(t) \in L_i(h)(t), \quad \text{a.e. },\ i = 1, 2.$$

Obviously

$$x^*(S(t_f,t)B(t,u_2(t)) < x^*(S(t_f,t)B(t,u_1(t)), \quad t \in C. \tag{27}$$

Now our desire is to substitute the measurable functions u_1 and u_2 by integrable ones, [30]. The substitution is realized in such a way keeping valid an inequality of the form (27). For $p > 0$ and $u \in S_U^1$ define

$$u_{1,p}(t) = \begin{cases} u_1(t), & \text{if } t \in C \text{ and } \|u_1(t)\| \leq p, \\ u(t), & \text{otherwise,} \end{cases}$$
$$u_{2,p}(t) = \begin{cases} u_2(t), & \text{if } t \in C \text{ and } \|u_2(t)\| \leq p, \\ u(t), & \text{otherwise.} \end{cases}$$

Then $u_{1,p}, u_{2,p} \in S_U^1$ and

$$x^*(S(t_f,t)B(t,u_{2,p}(t))) \leq x^*(S(t_f,t)B(t,u_{1,p}(t))), \quad t \in T. \tag{28}$$

For p sufficiently large the above inequality is strictly on a measurable subset having strictly positive measure. Let $x_{1,p}, x_{2,p}$ be the trajectories of system (22) corresponding to $u_{1,p}$, respectively $u_{2,p}$. Then for p sufficiently large we have

$$0 < \int_{t_0}^{t_f} x^*(S(t_f,t)[B(t,u_{1,p}(t)) - B(t,u_{2,p}(t))])\,dt = x^*(x_{1,p}(t_f) - x_{2,p}(t_f)).$$

Since the functional $x^* \in X^* \setminus \{0\}$ has been chosen arbitrary, we infer that the reachable set $R(t_f, 0)$ is not included in any closed hyperplane in X.

Hereafter the proof goes identically as the last part of proof to [30, theorem 2.1].

Now let us remove the assumption on the boundedness of the values of U. It means that M or m or both may be unbounded on some $t \in T$. Let us introduce the following functions

$$\overline{M}(t) = \begin{cases} M(t), & \text{if } M(t) < +\infty, \\ 1, & \text{otherwise,} \end{cases}$$
$$\overline{m}(t) = \begin{cases} m(t), & \text{if } m(t) > -\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then we define $\eta(t) := [\overline{M}(t) - \overline{m}(t)]/2$ and in (25) and (26) we consider $\overline{M}$ and $\overline{m}$ instead of M, respectively m. Then we repeat the last part of the proof of the previous case.

At the end we get the same conclusion on the reachable set as before. Thus the proof is complete. □

Remarks 5.1. (a) *If we consider the following scalar system*

$$x' = x + B(t,u),\ B(t,u) = u,\ u \in U := \mathbb{R}, \tag{29}$$

then following the estimation as in [30] *it results that $M = +\infty$ and $m = -\infty$. Thus $\eta = +\infty$ and the multifunctions L_1 and L_2 have empty values.*

(b) *Actually, the definitions of M and m as they appear in* [30] *are difficult to comprehend, since the supremum and the infimum are taken on a Hilbert space with no kind of order.*

(c) *The assumption "$\mu\{t \in T \mid U(t)$ is not a singleton$\} > 0$", as it is assumed in* [30]*, it is not enough, since if we consider again the scalar system of the following form*

$$x' = x + B(t,u),\ B(t,u) = f(t) \tag{30}$$

such that f is measurable and U is set-valued and measurable, arbitrary, then the reachable set is a singleton. Hence the study of controllability makes no sense.

(d) *As it is shown in* [30] *under some circumstances theorem 5.1 reduces to the Kalman criterion.*

(e) *When X is a separable real Hilbert space the above result has been obtained in* [27].

5.2. **The second result.** System (22) is said to be *approximately locally null controllable* if there exists an open neighborhood V of the origin such that for all $x_0 \in V$, $0 \in \mathrm{cl}(R(t_f, x_0))$.

Theorem 5.2.. *If $U : T \to \mathrm{CCo}(Y)$ is a weakly measurable multifunction such that for all $t \in T$, $U(t) \subset W$, where W is a weakly compact subset of Y and hypothesis (B) holds. Then*

(a) *if $S(t_f,t)B(t,U(t)) \neq \{0\}$ on a set of positive Lebesgue measure and (22) is approximately locally null-controllable, then there exists $x^* \in X^* \setminus \{0\}$ and $E \subset T$ Legesgue measurable such that*

$$\mu(E) > 0, \quad and \quad 0 < \sigma(x^*, S(t_f,t)B(t,U(t))) \quad \forall\, t \in E;$$

(b) *if $0 \in B(t,U(t)$ a.e. and for every $x^* \in X^* \setminus \{0\}$ there exists $E(x^*) \subset T$ Lebesgue measurable with $\mu(E(x^*)) > 0$ such that for all $t \in E(x^*)$ $0 < \sigma(x^*, S(t_f,t)B(t,U(t)))$, then (22) is approximately locally null controllable.*

6. Approximate null controllability of certain differential inclusions in infinite dimensional Banach spaces

This section is devoted to the study of the approximate null controllability of the following *Cauchy problem for quasi-differential inclusions*

$$\begin{cases} \dfrac{dx(t)}{dt} \in A(t,x(t))x(t) + F(t,x(t)), & \text{a.e. } t \in T \\ x(t_0) = x_0, \end{cases} \tag{31}$$

where $A(t,w)$ is a linear operator in a Banach space X depending on $t \in T$ and $w \in X$. If the operator A depends only on t, the differential inclusion (31) is said to be *semi-linear*.

Let T be the interval $T = [t_0, t_f]$, t_f fixed, and X a Banach space. A family of bounded linear operators $\mathcal{U}(t,s)$ on X, $t_0 \leq s \leq t \leq t_f$, depending on two parameters is said to be an *evolution system*, [31], if the following two conditions are fulfilled

(i) $\mathcal{U}(s,s) = I$, $\mathcal{U}(t,r)\mathcal{U}(r,s) = \mathcal{U}(t,s)$ for $t_0 \leq s \leq r \leq t \leq t_f$;
(ii) $(t,s) \mapsto \mathcal{U}(t,s)$ is strongly continuous for $t_0 \leq s \leq t \leq t_f$, i.e., $\lim_{t\downarrow s} \mathcal{U}(t,s)x = x$, for all $x \in X$.

We use the following assumptions

(X) X is a real separable Banach space;
(A) for every $u \in C(T,X)$ the family $\{A(t,u) \mid t \in T\}$ of linear operators generates a unique strongly continuous evolution system, $\mathcal{U}_u(t,s)$, $t_0 \leq s \leq t \leq t_f$;
(U) if $u \in C(T,X)$, the evolution system $\mathcal{U}_u(t,s)$, $t_0 \leq s \leq t \leq t_f$ satisfies:
  (i) $\exists\, c_1 \geq 0$ with $\|\mathcal{U}_u(t,s)\| \leq c_1$ for any $t_0 \leq s \leq t \leq t_f$, uniformly in u;
  (ii) $\exists\, c_2 \geq 0$ such that for any $u, v \in C(T,X)$ and any $w \in X$ there holds

$$\|\mathcal{U}_u(t,s)w - \mathcal{U}_v(t,s)w\| \leq c_2\|w\| \int_s^t \|u(\tau) - v(\tau)\|\, d\tau.$$

Remark 6.1. *If the operator A does not depend on w, but it depends on t, then the assumption (A) reads as follows: $\{A(t) \mid t \in I\}$ generates a unique strongly continuous evolution system $\mathcal{U}(t,s)$, $0 \leq s \leq t \leq T$. In this case we take $c_2 = 0$ (in (ii) from (U)).* $\triangle$

In connection with the multifunction F we will use the following assumptions:

(F_1) the multifunction F is defined on $T \times X$ and it has closed and nonempty values in X. For each $x \in X$, $F(\cdot\,, x)$ is measurable;
(F_2) F satisfies (F_1) and, moreover, it is $k(t)$-Lipschitz, i.e., there exits $k \in \mathcal{L}^1(T, \mathbb{R}_+)$ such that for almost all $t \in T$ and every $x,\ y \in X$, $\mathrm{D}(F(t,x),\, F(t,y)) \leq k(t)\|x - y\|$, D being the Hausdorff-Pompeiu metric;
(F_3) F is integrable bounded by a function $m \in \mathcal{L}^1(T, \mathbb{R}_+)$, i.e., for all $x \in C(T,X)$ and $t \in T$ we have that $F(t,x(t)) \subset m(t)B$, B being the unit closed ball in X;
(F_4) the function $T \ni t \mapsto \mathrm{d}(0,\, F(t,0))$ is integrable on I;
(F_5) $\mathrm{d}(0,\, F(t,0)) = 0$ on T.

Hereafter we are interested by the mild solutions of (31), i.e. the continuous functions having the following representation

$$x(t) = \mathcal{U}_x(t,t_0)x_0 + \int_{t_0}^{t_f} \mathcal{U}_x(t,s)f(s)\,ds, \quad t \in T, \quad f \in S_{F_x},$$

where S_{F_x} is the set of integrable selections from $F(\cdot, x(\cdot))$.

Together with (31) we consider

$$\text{(32)} \qquad \begin{cases} \dfrac{dy(t)}{dt} \in A(t, y(t))y(t) + g(t), \quad g \in \mathcal{L}^1(T, X), \quad \text{a.e. on } T \\ y(t_0) = y_0, \end{cases}$$

(S) Suppose that the problem (32) has the following mild solution

$$y(t) = \mathcal{U}_y(t,t_0)y_0 + \int_{t_0}^{t_f} \mathcal{U}_y(t,s)g(s)\,ds, \quad t \in T.$$

The system (31) is said to be *locally approximately null controllable* if for any $\varepsilon > 0$ there exists V a neighborhood of the origin and a solution x of (31) such that if $x(t_0) = x_0 \in V$, then $\|x(t_f)\| \leq \varepsilon$.

We admit that the assumptions (X), (A), (F_2), and (F_3) are fulfilled and we consider the problems (31) and (32). Let us denote $\delta = \|x_0 - y_0\|$, $p = c_2(\|x_0\| + \|m\|_1)$, $K(t) = \int_o^t [p + 2c_1 k(s)]ds$, $E(t) = \exp(K(t))$, $t \in I$. Moreover, we invoke (S) and denote $\gamma(t) = \mathrm{d}(g(t), F(t, y(t)))$, $t \in T$. Based on Lemma 2.3 in [23] or on Lemma 2.15 in [28] we have that $\gamma \in \mathcal{L}^1$ and so let $n(t) = c_1[\delta + \int_0^t 2\gamma(s)\,ds]$, $t \in T$. The following Filippov type existence result takes place.

Theorem 6.1.. ([23]) *We admit the following assumptions* (X), (A), (U), (F_2), (F_3), (F_4), *and* (S). *Then problem* (31) *has a mild solution* $x \in C(I,\, X)$ *such that*

$$\text{(33)} \qquad \|x(t) - y(t)\| \leq c_1 \left[\delta E(t) + \int_{t_0}^{t} \frac{E(t)}{E(s)} 2\gamma(s)\,ds\right], \quad \textit{on } T$$

and

$$\text{(34)} \quad \|f(t) - g(t)\| \leq 2\gamma(t) + 2k(t)c_1 \left[\gamma(t)E(t) + \int_{t_0}^{t} \frac{E(t)}{E(s)} 2\gamma(s)\,ds\right], \quad \textit{a.e. on } T.$$

It results the next continuous dependence result

Theorem 6.2.. ([23]) *Let* $f, g \in \mathcal{L}^1(T, X)$, $\chi = \|f - g\|_1$, *such that there are satisfied all the assumptions of the Theorem 6.1 taking* f *instead of* F. *Denote by* x *and* y *two mild solutions of the quasi-linear equations corresponding to* f, x_0, *respectively* g, y_0. *Then the following estimation holds*

$$\|x(t) - y(t)\| \leq c_1(\chi + \delta)\exp[c_2(\min\{\|x_0\|,\ \|y_0\|\} + \min\{\|f\|_1,\ \|g\|_1\})t], \quad t \in T.$$

Now it comes the approximately null controllability result

Theorem 6.3.. *Suppose that there fulfilled all the assumptions of Theorem 6.1 and, moreover,* (F_5). *Then the problem* (31) *is locally approximately null controllable.*

Proof. In (32) we consider $g = 0$ and $y_0 = 0$. Then $\gamma = 0$. From Theorem 6.2 we have that the unique mild solution of (32) is $y(t) = 0$ on T. Applying Theorem 6.1 from (33) we get that

$$\|x(t_f) - 0\| \leq c_1 \delta E(t_f). \tag{35}$$

But $E(t_f)$ is bounded, hence if $\delta = \|x_0\|$ is small enough, then the right hand side of (35) is as small as we wish. □

Remark 6.2. *In our approach the assumption* (F_5) *is necessary, too. Let us consider the following example*

$$dz(t)/dt = z + z^2 + 1/4, \quad t \geq 0, \quad z(0) = z_0. \tag{36}$$

The general solution of the equation (36) *is*

$$z(t) = -1/2 + 1/[2/(2z_0 + 1) - t]. \tag{37}$$

The function defined by (37) *for* z_0 *in absolute value small does not converge to* 0 *for any positive* t. *Moreover, it remains far away from zero. Here* $A = 1$. $F(t, z(t))$ *is equal to* $z^2(t) + 1/4$ *and does not satisfy* (F_5). △

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