

## VARIATIONAL INEQUALITIES, COMPLEMENTARITY PROBLEMS AND PSEUDO-MONOTONICITY. DYNAMICAL ASPECTS

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**Abstract.** In this paper we will present several existence theorems and we will study the stability of equilibrium given by a variational inequality or by a complementarity problem in a general Hilbert space, using the notion of local projected dynamical system.

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### 1. INTRODUCTION

It is well known that *variational inequalities* are systematically used in the theory of many practical problems related to "equilibrium".

The equilibrium is an important state considered in Physics, Engineering, Sciences and Economics and even in Biology, [5], [37], [38], [24], [56]. When a variational inequality is defined on a closed convex set, which is in particular a closed convex cone, in this case our variational inequality is a *complementarity problem*. We have this situation in many problems considered in Economics. Because of this, the Complementarity Theory is strongly related to the study of equilibrium [16], [24], [36], [37], [38], [49].

The problem to find the equilibrium of an economical system is exactly the problem to find the solutions of a variational inequality or of a complementarity problem.

It is known that the solvability of a variational inequality or of a complementarity problem is not an evident problem. Because of this fact, there exist many existence theorems [5], [11], [16], [36]-[49], [56], [73]-[77].

The solutions of a variational inequality or a complementarity problem give us a static information about an equilibrium state. In many problems in Economics, in Mechanics, or in Engineering we are interested to know *the evolution* of equilibrium with respect to the parameter "time".

Certainly, we must study the evolution of equilibrium under the constraints used in the definition of the set with respect to which the variational inequality or the complementarity problem is considered.

This fact is now possible by using the notion of "local projected dynamical system", defined in [22] and studied in [23], [63], [57]-[63], [72] and [21].

When we associate to a variational inequality, or to a complementarity problem, a local projected dynamical system, the critical points (i.e. the equilibrium points) of this dynamical system are exactly the solutions of the variational inequality (respectively of the complementarity problem) and therefore are the equilibrium points of considered practical problems.

Obviously, by this method, we can apply the theory of dynamical systems to the study of variational inequalities or to the study of complementarity problems from the dynamical point of view.

We will show that the pseudo-monotonicity plays an important role. The paper will be finished with some comments and open problems.

## 2. PRELIMINARIES

We denote by  $(H, \langle \cdot, \cdot \rangle)$  a Hilbert space and by  $(R^n, \langle \cdot, \cdot \rangle)$  the  $n$ -dimensional Euclidean space. The closed pointed convex cone of non-negative real numbers will be denoted by  $R_+$ , i.e.  $R_+ = \{\alpha \in R \mid \alpha \geq 0\}$ .

We say that a non-empty subset  $K$  of a Hilbert space  $H$  is a *convex cone* if the following properties are satisfied:

( $k_1$ ).  $K + K \subseteq K$  and

( $k_2$ ).  $\lambda K \subseteq K$ , for every  $\lambda \in R_+$ .

If the set  $K$  satisfies also the property

( $k_3$ ).  $K \cap (-K) = \{0\}$

then, in this case we say that  $K$  is a *pointed convex cone*. Generally, we will suppose in this paper that  $K$  is a closed pointed convex cone or only a closed convex set.

If  $K$  is a pointed convex cone in  $H$ , the dual cone of  $K$  is, by definition  $K^* = \{y \in H \mid \langle x, y \rangle \geq 0, \text{ for all } x \in K\}$ . It is known that for any pointed convex cone  $K$ , the dual  $K^*$  is always a closed convex cone. The polar cone of  $K$  is  $K^\circ = -K^*$ .

Let  $D$  be a subset of  $H$  and  $x \in \bar{D}$  (the closure of  $D$ ). If  $u$  is an element in  $H$ , the distance from  $u$  to the set  $D$  will be denoted by  $d(u; D)$ . The *contingent cone* of Bouligand at the point  $x$  is defined as

$$T_D(x) := \{v \in H \mid \liminf_{t \rightarrow 0^+} \frac{d(x + tv; D)}{t} = 0\}.$$

It is known that  $T_D(x)$  is always closed and is included in the cone  $\overline{\bigcup_{t>0} \frac{1}{t}(D - x)}$ .

The Bouligand cone is not necessarily convex nor is the map  $x \rightarrow T_D(x)$  generally closed, nor lower semicontinuous. This lack of regularity of the cone-valued map  $T_D(\cdot)$  makes it inadequate for the study of many problems where tangency conditions are essential.

A smaller convex cone, known as the *Clarke cone*, and denoted by  $C_D(x)$ , is in many instances more appropriate. The cone  $C_D(x)$  is a closed convex cone and one of its definitions is  $C_D(x) := \{v \in H \mid \forall t_n \rightarrow 0^+ \text{ and } \forall x_n \rightarrow x \text{ in } D, \exists v_n \rightarrow v \text{ such that } x_n + t_n v_n \in D, \forall n\}$ .

The inclusion  $C_D(x) \subseteq T_D(x) \subseteq \overline{\bigcup_{t>0} \frac{1}{t}(D-x)}$  always holds, [2].

When the first inclusion is an equality, the set  $D$  is said to be *tangentially regular at  $x$*  (in fact  $C(\cdot) = T_D(\cdot)$  at points where the set-valued map  $T_D(\cdot)$  is lower semicontinuous).

**REMARK.** For convex sets we have  $C_D(x) = T_D(x) = \overline{\bigcup_{t>0} \frac{1}{t}(D-x)}$  and for interior points  $x$ , all three cones cover the whole space  $H$ .

Let  $D$  be a non-empty closed convex subset of  $H$  and  $x \in D$ . The *normal cone*  $N_D(x)$  to  $D$  at  $x$  is defined by

$$N_D(x) := \{p \in H^*(= H) \mid \langle p, x \rangle = \max_{y \in D} \langle p, y \rangle\},$$

or equivalently

$$\begin{aligned} N_D(x) &:= \{p \in H^*(= H) \mid \langle p, x - y \rangle \geq 0, \text{ for all } y \in D\} \\ &= \{p \in H^*(= H) \mid \langle p, y - x \rangle \leq 0, \text{ for all } y \in D\}. \end{aligned}$$

It is known that  $N_D(x) = (T_D(x))^o := -(T_D(x))^*$ . Finally, we need to recall some results about the projection operator onto a closed convex set.

Let  $D$  be a closed convex set in a Hilbert space  $(H, \langle \cdot, \cdot \rangle)$ . For any  $x \in H$ , there exists a unique element  $P_D(x) \in D$  such that  $\|x - P_D(x)\| \leq \|x - y\|$ , for all  $y \in D$ , [5], [36].

**PROPOSITION 1.** *If  $D \subset H$  is a non-empty closed convex set and  $x \in H$  is an arbitrary element, then the following statements are equivalent:*

- (1)  $\|x - P_D(x)\| \leq \|x - y\|$ , for all  $y \in D$ ,
- (2)  $\langle x - P_D(x), P_D(x) - y \rangle \geq 0$ , for all  $y \in D$ .

Moreover, for each  $x \in H$ ,  $y = P_D(x)$  if and only if  $x \in y + N_D(y)$ .

*Proof.* This is a classical result and for a proof, the reader is referred to [5] or [36] and for the last conclusion to [2].  $\square$

If  $D = K$  where  $K$  is a closed convex cone in  $H$ , then in this case we have the following result.

**PROPOSITION 2.** *For any  $x \in H$ , the projection  $P_K(x)$  is characterized by the following two properties:*

- (1)  $\langle P_K(x) - x, y \rangle \geq 0$ , for all  $y \in K$ ,
- (2)  $\langle P_K(x) - x, P_K(x) \rangle = 0$ .

*Proof.* The reader is referred to [36] for a proof of this result.  $\square$

### 3. VARIATIONAL INEQUALITIES AND COMPLEMENTARITY PROBLEMS.

In the study of equilibrium and in particular, in the study of economical equilibrium *variational inequalities* and *complementarity problems* are systematically used. In this sense the reader is referred to [5], [10], [53], [56], [47], [63], [73]-[74] and [77] for variational inequalities and to [11]-[12], [16], [24], [36]-[46], [48]-[50], [75] and [77] for complementarity problems.

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $D \subset H$  a closed convex set and  $K \subset H$  a closed convex cone.

Given a mapping  $f : H \rightarrow H$ , the variational inequality defined by the mapping  $f$  and the set  $D$  is

$$VI(f, D) := \left\{ \begin{array}{l} \text{find } x_0 \in D \text{ such that} \\ \langle y - x_0, f(x_0) \rangle \geq 0, \text{ for all } y \in D \end{array} \right.$$

and the complementarity problem defined by  $f$  and the cone  $K$  is

$$CP(f, K) := \left\{ \begin{array}{l} \text{find } x_0 \in K \text{ such that} \\ f(x_0) \in K^* \text{ and } \langle x_0, f(x_0) \rangle = 0 \end{array} \right. .$$

It is well known (see [37], [38]) that when  $D = K$ , we have that  $VI(f, K)$  and  $CP(f, K)$  are equivalent problems.

Generally, the theory of complementarity problems is different from the theory of variational inequalities, since for a complementarity problem the set  $K$  is unbounded and we have also the ordering defined by  $K$ . The classical theory of variational inequalities has been developed, generally, for bounded (or compact) convex sets.

We note that many authors obtained recently interesting results for variational inequalities, in the case when the set  $D$  is unbounded, using KKM mappings or generalizations of KKM mappings.

For the theory of variational inequalities the reader is referred to [5], [53], among others and for complementarity theory the monographs [37], [38] are enough.

The mathematical models for many problems related to equilibrium generally are variational inequalities or complementarity problems.

### 4. PSEUDO-MONOTONE AND QUASI-MONOTONE OPERATORS

We intend to present in this paper some stability theorems for solutions to variational inequalities and complementarity problems with pseudo-monotone or quasi-monotone operators.

We will use the pseudo-monotonicity defined in 1976 by S. Karamardian [50]. Generalizations of monotonicity are now considered by many authors in these papers: [17]-[20], [29], [31]-[33], [50]-[52], [55], [64]-[65].

As generalizations of monotonicity we will consider the pseudo-monotone operators and quasi-monotone operators. Any monotone operator is pseudo-monotone and any pseudo-monotone operator is quasi-monotone.

Other generalizations of monotonicity are between monotone and pseudo-monotone or between pseudo-monotone and quasi-monotone. Now, we recall the definitions.

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $D \subset H$  a non-empty subset. We recall that a mapping  $f : H \rightarrow H$  is *monotone* on  $D$  if for any  $x_1, x_2 \in D$  we have  $\langle x_1 - x_2, f(x_1) - f(x_2) \rangle \geq 0$ .

If  $f : H \rightarrow H$  is a set-valued mapping, we say that  $f$  is *monotone* if for any  $x_1, x_2 \in D$  and any  $y_1 \in f(x_1), y_2 \in f(x_2)$  we have  $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ .

We say that a mapping  $f : H \rightarrow H$  is *pseudo-monotone* on  $D$  if for any  $x, y \in D$  we have that  $\langle y - x, f(x) \rangle \geq 0$  implies  $\langle y - x, f(y) \rangle \geq 0$ .

Obviously, any monotone mapping is pseudo-monotone, while the converse is not true [50], [51].

The pseudo-monotonicity can be extended to set-valued mappings. Indeed, if  $f : H \rightarrow H$  is a set-valued mapping, we say that  $f$  is *pseudo-monotone* on  $D$  if for any  $x_1, x_2 \in D$  and any  $x_1^* \in f(x_1), x_2^* \in f(x_2)$  we have that  $\langle x_1^*, x_2 - x_1 \rangle \geq 0$  implies  $\langle x_2^*, x_2 - x_1 \rangle \geq 0$ .

We say that a mapping  $f : H \rightarrow H$  is strongly *pseudo-monotone* on  $D$  if there exists  $\beta > 0$  such that for each pair of distinct points  $x, y \in H$ , we have that  $\langle y - x, f(x) \rangle \geq 0$  implies  $\langle y - x, f(y) \rangle \geq \beta \|x - y\|^2$ .

A mapping  $f : H \rightarrow H$  is *quasi-monotone* on  $D$  if for every pair of distinct points  $x, y \in D$  we have that  $\langle y - x, f(x) \rangle > 0$  implies  $\langle y - x, f(y) \rangle \geq 0$ . It is known [51] that any pseudo-monotone mapping is quasi-monotone, but the converse is not true.

Suppose that  $H = R^n$  endowed with the Euclidian structure and the set  $D = R_+^n$ .

Given a real  $n \times n$ -matrix  $M$  and an arbitrary vector  $q \in R^n$  we have that the affine mapping  $\phi(x) = Mx + q$  is pseudo-monotone on  $R_+^n$  if  $x, y \in R_+^n$  and  $\langle y - x, Mx + q \rangle \geq 0$  imply that  $\langle y - x, My + q \rangle \geq 0$ .

In this case, if there exists a vector  $x_* \in R_+^n$  such that  $Mx_* + q \in R_+^n$  and the mapping  $\phi(x) = Mx + q$  is pseudo-monotone then the matrix  $M$  is *copositive* (i.e.  $\langle x, Mx \rangle \geq 0$  for all  $x \in R_+^n$ ) and belongs to  $\mathcal{P}_0$  (i.e. every principal minor of  $M$  is nonnegative). A proof of this result is in [29]. We recall also the following result. A matrix  $M$  is *positive semidefinite* if and only if for every  $q \in R^n$ , the affine mapping  $\phi(x) = Mx + q$  is pseudo-monotone [29].

Let  $f : R^n \rightarrow R$  be a differentiable function on an open set  $D \subset R^n$ . We say that  $f$  is *pseudoconvex* on  $D$  if for every pair of distinct points  $x, y \in D$  we have  $\langle y - x, \nabla f(x) \rangle \geq 0 \implies f(y) \geq f(x)$ .

Also we say that  $f$  is *quasiconvex* on  $D$  (now,  $D$  is supposed to be a convex set), if for all  $x, y \in D$  and all  $\lambda \in [0, 1]$  we have  $f(y) \leq f(x) \implies f(\lambda x - (1 - \lambda)y) \leq f(x)$ .

The following proposition is a classical result.

**PROPOSITION 3.** *Let  $f$  be a real-valued differentiable function on an open convex set  $D \subset R^n$ . Then  $f$  is quasiconvex on  $D$  or  $f$  is pseudoconvex on  $D$  if and only if  $\nabla f$  is quasi-monotone on  $D$  or  $f$  is pseudo-monotone on  $D$ . (We denoted by  $\nabla f$  the gradient of  $f$ .)*

*Proof.* For a proof of this result, the reader is referred to [52] and [65]. □

Other tests for pseudo-monotonicity and quasi-monotonicity are presented in [18]. For other results on pseudo-monotone and quasi-monotone operators, the reader is referred to [17]-[20], [29], [31]-[33], [51]-[52] and [64]-[65].

5. ABOUT THE SOLVABILITY OF VARIATIONAL INEQUALITIES AND  
COMPLEMENTARITY PROBLEMS ON UNBOUNDED SETS

Generally, in many economic problems, we need to study the equilibrium of an economic system, with respect to an unbounded convex set. In many situations, this set is a closed convex set which is, in particular, a closed convex cone or a closed polyhedral convex set.

Therefore, we need to study the existence of solutions of a variational inequality or of a complementarity problem, defined on an unbounded closed convex set. The study of solvability of variational inequalities, defined on compact or on bounded convex sets, is the subject considered in the classical theory of variational inequalities.

In this section, we will study the solvability of variational inequalities defined on *unbounded, closed and convex sets*, considering two situations:

- (I). the mapping  $f$  is  $\phi$ -asymptotically bounded and  
(II). the mapping  $f$  is without exceptional family of elements.**

We will consider each situation separately.

- (I). The mapping  $f$  is  $\phi$ -asymptotically bounded.**

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $D \subset H$  a closed unbounded set and  $f : H \rightarrow H$  a set-valued mapping.

**DEFINITION 1.** *We say that  $f$  is  $\phi$ -asymptotically bounded if there exist two positive real numbers  $r$  and  $\beta$  such that the conditions  $x \in D$  and  $r \leq \|x\|$  imply  $\|y\| \leq \beta\phi(\|x\|)$ , for all  $y \in f(x)$ , where  $\phi$  is a continuous function from  $R_+$  into  $R_+$  such that for some  $\alpha > 0$ ,  $\phi(t) > 0$  for  $t \geq \alpha$ .*

Obviously, Definition 1 can be applied to single-valued mappings.

**REMARK.** The notion of *single-valued,  $\phi$ -asymptotically bounded operator* was introduced in 1984 by V. H. Weber. In [70], Weber studied the  $\phi$ -asymptotically bounded operators from the point of view of the spectral analysis. The notion of  $\phi$ -asymptotically bounded operator has interesting applications in complementarity theory [37], [38].

Before giving the next existence theorem for variational inequalities, we need to recall some notions and results, well-known in the general topology and in Nonlinear Analysis. We will recall the Darbo's Fixed Point Theorem for  $k$ -set contractions and the Eilenberg-Montgomery's Fixed Point Theorem for set-valued mappings.

Let  $A$  be a subset of  $H$ . The *Kuratowski measure of non-compactness* of  $A$  is by definition  $\alpha(A) = \inf\{\epsilon > 0 \mid A \text{ can be covered by a finite number of sets of diameter less than } \epsilon\}$ .

Let  $\Omega \subset H$  be a subset and  $f : \Omega \rightarrow H$  a mapping. We say that  $f$  is a  *$k$ -set contraction* if the following conditions are satisfied:

- (1)  $f$  is continuous, bounded (i.e. for any bounded set  $B$ ,  $f(B)$  is bounded) and
- (2) there exists a real number  $k \in [0, 1[$  such that, for each bounded subset  $A \subset \Omega$ , we have  $\alpha(f(A)) \leq k\alpha(A)$ .

It is known that  $\alpha(A) = 0$  if and only if  $A$  is a relatively compact set. Any completely continuous mapping is a  $k$ -set contraction, but the converse is not true. For more details about the measure of non-compactness, the reader is referred to [1].

**THEOREM 4.** [DARBO] *If  $\Omega \subset H$  is a non-empty, closed, bounded and convex set and  $f : \Omega \rightarrow \Omega$  is a  $k$ -set contraction, then  $f$  has a fixed point in  $\Omega$ .*

*Proof.* For a proof of this classical result, the reader is referred to [79]. □

We say that a mapping  $f : H \rightarrow H$  is a  $k$ -set field if  $f$  has a representation of the form  $f(x) = x - T(x)$ , where  $T : H \rightarrow H$  is a  $k$ -set contraction.

Let  $X$  be a topological space and  $M \subset X$  a non-empty set. We say that a continuous mapping  $r : X \rightarrow M$  is a *retraction* if and only if  $r(x) = x$ , for all  $x \in M$ . In this case we say that  $M$  is a *retract* of  $X$ .

A set  $D$  in the topological space  $X$  is called a *neighborhood retract* if and only if  $D$  is a retract of some of its neighborhoods  $U$ . We recall also that an ANR (*absolute neighborhood retract*) space is a compact metric space  $M$  with the universal property that every homeomorphic image of  $M$ , in a separable metric space is a neighborhood retract set. A prototype of an ANR space is a compact convex set in a Banach space.

A subset  $D$  of  $H$  is called *contractible* if there is a continuous mapping  $h : D \times [0, 1] \rightarrow D$  such that  $h(x, 0) = x$  and  $h(x, 1) = x_0$ , for some  $x_0 \in D$ .

We note that if  $D$  is convex, it is contractible, since for any  $x_0 \in D$ , the mapping  $h(x, t) = tx_0 + (1 - t)x$ , where  $t \in [0, 1]$  and  $x \in D$ , would satisfy the above property. Also, a set star-shaped at  $x_0$  is contractible to  $x_0$ .

Consider again a topological space  $X$ . Denote by  $H_n(X)$  the  $n$ -dimensional singular homology group of the space  $X$ . We recall that a topological space  $X$  is *acyclic* if  $H_n(X) = 0$ , for any  $n > 0$ . It is known that any contractible topological space is acyclic. A compact convex set in a Hilbert space is acyclic and an ANR space.

Let  $X$  and  $Y$  be topological spaces and let  $f : X \rightarrow Y$  be a set-valued mapping. We recall that  $f$  is *upper semicontinuous* on  $X$  if the set  $\{x \in X \mid f(x) \subset V\}$  is open in  $X$ , whenever  $V$  is an open subset in  $Y$ , [3].

**THEOREM 5.** [EILENBERG-MONTGOMERY] *Let  $M$  be an acyclic ANR space and  $T : M \rightarrow M$  an upper semicontinuous set-valued mapping such that for every  $x \in M$ , the set  $T(x)$  is acyclic. Then  $T$  has a fixed point (i.e. there exists  $x_* \in M$  such that  $x_* \in T(x_*)$ ).*

*Proof.* See S. Eilenberg and D. Montgomery: *Fixed point theorems for multi-valued transformations*, Amer. J. Math., 68(1946), 214-222. □

In a general Hilbert space we have the following existence theorem.

**THEOREM 6.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $D \subset H$  an unbounded, closed convex set and  $f : H \rightarrow H$  a  $k$ -set field with the representation  $f(x) = x - T(x)$ , for all  $x \in H$ . If the following assumptions*

- (1)  *$T$  is  $\phi$ -asymptotically bounded on  $D$ ,*
- (2)  *$\lim_{t \rightarrow +\infty} \phi(t)$  is finite*

*are satisfied, then the problem  $VI(f, D)$  has a solution.*

*Proof.* From the theory of variational inequalities, we know that the problem  $VI(f, D)$  has a solution if and only if the mapping  $\psi : H \rightarrow H$  defined by

$$\psi(x) = P_D[x - f(x)] = P_D[T(x)]$$

for all  $x \in H$ , has a fixed point. Considering the properties of  $k$ -set contractions, we can show that  $\psi$  is also a  $k$ -set contraction. Now we prove that  $\psi(D)$  is bounded. Indeed, since  $T$  is  $\phi$ -asymptotically bounded then for any  $x \in D$  with  $r \leq \|x\|$  we have  $\|\psi(x)\| = \|P_D[T(x)]\| \leq \|T(x)\| \leq \beta\phi(\|x\|)$ , which implies that  $\psi$  is  $\phi$ -asymptotically bounded. Therefore, since  $\psi$  is  $\phi$ -asymptotically bounded and  $\lim_{t \rightarrow \infty} \phi(t) < +\infty$ , there exist  $\delta > 0$  and  $h > 0$  such that  $\|\psi(x)\| \leq \delta$ , for all  $x \in D$  with  $\|x\| > h$ .

Obviously,

$$\psi(D) = \{\psi(x) \mid x \in D, \|x\| > h\} \cup \{\psi(x) \mid x \in D, \|x\| \leq h\}.$$

So it suffices to show that the set  $\{\psi(x) \mid x \in D, \|x\| \leq h\}$  is bounded. But this follows from the fact that  $\psi$  is a  $k$ -set contraction. Let us define the set

$$B := cl(conv \psi(D)).$$

Since  $D$  is closed and convex and  $\psi(D) \subseteq D$  we have that  $B \subseteq D$ . Therefore  $\psi(B) \subseteq \psi(D)$  and hence by the definition of  $B$ , we obtain that  $\psi(B) \subseteq B$ .

Obviously,  $B$  is closed, convex and bounded. Applying the Darbo Fixed Point Theorem to the  $k$ -set contraction  $\psi|_B$  and to the set  $B$  we obtain that  $\psi$  has a fixed point, which implies that the problem  $VI(f, D)$  has a solution.  $\square$

The next result is a variant of Theorem 6 for set-valued mappings in the Euclidean space  $(R^n, \langle \cdot, \cdot \rangle)$ .

Let  $D \subseteq R^n$  be a closed convex unbounded set and  $f : R^n \rightarrow R^n$  a set-valued mapping.

In this case we consider the following multivalued variational inequality

$$MVI(f, D) : \begin{cases} \text{find } x_* \in D \text{ such that} \\ \text{there exists } y_* \in f(x_*) \text{ with} \\ \langle u - x_*, y_* \rangle \geq 0, \text{ for all } u \in D \end{cases}$$

We have the following result.

**THEOREM 7.** *Let  $(R^n, \langle \cdot, \cdot \rangle)$  be the  $n$ -dimensional Euclidean space,  $D \subseteq R^n$  an unbounded closed, convex set and  $f : R^n \rightarrow R^n$  a set-valued mapping such that  $f(x) = x - T(x)$ , where  $T : R^n \rightarrow R^n$  is an upper semicontinuous mapping.*

*If the following assumptions are satisfied*

- (1)  $T$  is  $\phi$ -asymptotically bounded,
- (2)  $T$  is bounded (i.e. for any bounded set  $B \subseteq R^n$ ,  $T(B)$  is bounded),
- (3) for any  $x \in D$ , the set  $T(x)$  is a closed contractible set,
- (4)  $\lim_{t \rightarrow +\infty} \phi(t)$  is finite,

*then the problem  $MVI(f, D)$  has a solution.*

*Proof.* We consider the set-valued mapping  $\psi : R^n \rightarrow R^n$ , defined by

$$\psi(x) = P_D[x - f(x)] = P_D[T(x)].$$

The mapping  $\psi$  is upper semicontinuous and any fixed point of  $\psi$  (i.e.,  $x_* \in \psi(x_*)$ ) is a solution of the problem  $MVI(f, D)$ .



For any  $x \in D$ , the set  $\psi(x)$  is contractible (and hence acyclic). Indeed, if  $x \in D$  is an arbitrary element, denote by  $A = T(x)$ . By assumption,  $A$  is contractible, i.e. there exists a continuous function  $h : A \times [0, 1] \rightarrow A$  with the following properties

- (1)  $h(u, 0) = u$ , for all  $u \in A$ ,
- (2)  $h(u, 1) = u_0$ , for all  $u \in A$  and some  $u_0 \in A$ .

Consider the function  $h^* : P_D[A] \times [0, 1] \rightarrow P_D[A]$ , defined by  $h^*(P_D(u), \lambda) := P_D[h(u, \lambda)]$ , for all  $P_D(u) \in P_D[A]$  and all  $\lambda \in [0, 1]$ .

We have  $h^*(P_D(u), 0) = P_D(h(u, 0)) = P_D(u)$  and  $h^*(P_D(u), 1) = P_D(h(u, 1)) = P_D(u_0)$ .

We must show also that  $h^*$  is continuous. Indeed, let  $\{(P_D(u_n), \lambda_n)\}_{n \in \mathbb{N}}$  be a sequence in  $P_D[A] \times [0, 1]$ , convergent to an element  $(P_D(u_*), \lambda_*)$ .

Because  $A$  is closed and bounded, we have that  $A \times [0, 1]$  is compact. Then there exists a subsequence  $\{u_{n_k}, \lambda_{n_k}\}_{k \in \mathbb{N}}$  of the sequence  $\{u_n, \lambda_n\}_{n \in \mathbb{N}}$  convergent to an element  $(v, \lambda_*) \in A \times [0, 1]$ . Using the continuity of  $P_D$  and the uniqueness of the limit, we have

$$\lim_{k \rightarrow \infty} P_D(u_{n_k}) = P_D(v) = P_D(u_*).$$

We deduce that

$$\begin{aligned} \lim_{n \rightarrow \infty} h^*(P_D(u_n), \lambda_n) &= \lim_{k \rightarrow \infty} h^*(P_D(u_{n_k}), \lambda_{n_k}) \\ &= \lim_{k \rightarrow \infty} P_D[h(u_{n_k}, \lambda_{n_k})] = P_D[h(v, \lambda_*)] \\ &= h^*(P_D(v), \lambda_*) = h^*(P_D(u_*), \lambda_*). \end{aligned}$$

Since  $\psi$  is also  $\phi$ -asymptotically bounded and because for any bounded set  $S \subset D$ , we have that  $P_D[T(S)]$  is bounded, we can show (following the proof of Theorem 6) that  $\psi(D)$  is bounded. Now, we consider the set

$$B := cl(conv \psi(D)),$$

which is closed convex compact and hence it is an acyclic ANR space. As in the proof of Theorem 6, we have that  $\psi(B) \subseteq B$ .

We observe that Theorem 5 [Eilenberg-Montgomery] is applicable to the set  $B$  and to the set-valued mapping  $\psi|_B$  and therefore  $\psi$  has a fixed point, which implies that the problem  $MVI(f, D)$  has a solution.  $\square$

**(II). The mapping  $f$  is without exceptional families of elements.**

Recently, (see [43],[11]) the first author introduced a new topological method in the study of solvability of complementarity problems and of variational inequalities. This method is based on the notion of *exceptional family of elements* of a mapping, which is supported by the Leray-Schauder type alternatives or by the topological degree. Many results have been shown by this method [11],[40]-[49], [73]-[77].

First, in this section we will consider the case of complementarity problems. For this case we will recall a general result obtained recently by this method and we will add two new results.

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $K \subset H$  a closed pointed convex cone and  $f : H \rightarrow H$  a mapping. Consider the problem  $NCP(f, K)$  defined by  $f$  and  $K$ .

**DEFINITION 2** (42). *We say that  $\{x_r\}_{r>0} \subset H$  is an exceptional family of elements for the mapping  $f(x)$  with respect to  $K$  if for any real number  $r > 0$  there exists a real number  $\mu_r > 1$  such that the vector  $u_r = \mu_r x_r + f(x_r)$  satisfies the following conditions:*

- (1)  $u_r \in K^*$ ,
- (2)  $\langle \mu_r, x_r \rangle = 0$ ,
- (3)  $\|x_r\| \rightarrow +\infty$  as  $r \rightarrow +\infty$ .

For the next theorem we need to recall the following notions. Let  $f : H \rightarrow H$  be a mapping. We say that  $f$  is a *completely continuous field* if  $f$  has a representation of the form  $f(x) = x - T(x)$ , where  $T : H \rightarrow H$  is a completely continuous mapping, that is,  $T$  is continuous and for any bounded set  $B \subset H$ ,  $T(B)$  is relatively compact. Let  $\alpha$  be the Kuratowski measure of non-compactness [1], [6]. We say that a mapping  $T : H \rightarrow H$  is  $\alpha$ -*condensing* if  $T$  is continuous, bounded and  $\alpha(T(B)) < \alpha(B)$ , for all  $B \subset H$  such that  $\alpha(B) > 0$ .

We say that a mapping  $f : H \rightarrow H$  is an  $\alpha$ -*condensing field* if  $f$  has a representation of the form  $f(x) = x - T(x)$ , where  $T : H \rightarrow H$  is an  $\alpha$ -condensing mapping.  $T : H \rightarrow H$  is *pseudo-contractant* if the mapping  $f(x) = x - T(x)$  is monotone. We say that  $f : H \rightarrow H$  is *projectionally pseudo-contractant* with respect to  $K$  if the mapping  $\phi(x) = P_K[x - f(x)]$  is pseudo-contractant. It is known that if  $f$  is a *non-expansive field*, i.e.  $f(x) = x - T(x)$ , where  $T : H \rightarrow H$  is a non-expansive mapping, then  $f$  is *projectionally pseudo-contractant*.

We have the following alternative theorem for complementarity problems.

**THEOREM 8.** [42] *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $K \subset H$  a closed pointed convex cone and  $f : H \rightarrow H$  a mapping. If  $f$  satisfies one of the following conditions:*

- (1)  $f$  is a completely continuous field,
- (2)  $f$  is an  $\alpha$ -condensing field,
- (3)  $f$  is a projectionally pseudo-contractant field,

*then there exists either a solution to the problem  $NCP(f, K)$ , or  $f$  has an exceptional family of elements with respect to  $K$ .*

*Proof.* A proof of this result is given in [42]. □

**COROLLARY 9.** *If a mapping  $f : H \rightarrow H$  is continuous, satisfies one of assumptions of Theorem 8 and it is without exceptional family of elements with respect to  $K$ , then the problem  $NCP(f, K)$  has a solution.*

In our papers [11],[38]-[49] and in [73]-[77] are presented several classes of mappings without exceptional families of elements. For the next two new existence results we need to recall the following definition.

**DEFINITION 3.** [39], [42] *We say that a mapping  $f : H \rightarrow H$  satisfies condition  $(\theta)$  with respect to a convex cone  $K \subset H$  if there exists  $\rho > 0$  such that for each  $x \in K$  with  $\|x\| > \rho$ , there exists  $y \in K$  with  $\|y\| < \|x\|$  such that*

$$\langle x - y, f(x) \rangle \geq 0.$$

In [39] it is proved that, if  $f$  satisfies condition  $(\theta)$  with respect to  $K$ , then  $f$  is without exceptional family of elements with respect to  $K$ .

As applications of condition  $(\theta)$  we will give the following two new existence theorems for complementarity problems.

**THEOREM 10.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $f : H \rightarrow H$  a completely continuous field, or an  $\alpha$ -condensing field or a projectionally pseudo-contractant field. If there exists a non-empty bounded set  $D \subset K$  such that the set*

$$M = \bigcap_{y \in D} \{x \in K \mid \|f(x)\| < \|y - x + f(x)\|\}$$

*is bounded or empty then the problem  $NCP(f, K)$  has a solution.*

*Proof.* Since the sets  $D$  and  $M$  are bounded, there exists  $\rho_0 > 0$  such that  $M \cup D \subset \{x \in K \mid \|x_0\| \leq \rho_0\}$ . If  $x \in K$  is an arbitrary element such that  $\rho_0 < \|x\|$ , then we have that  $x \notin M$ , which implies that for some  $y_0 \in D$  we have  $\|f(x)\| \geq \|y_0 - x + f(x)\|$ , or  $\|f(x)\|^2 \geq \|y_0 - x + f(x)\|^2$  and finally

$$\langle f(x), f(x) \rangle \geq \langle y_0 - x + f(x), y_0 - x + f(x) \rangle,$$

which implies

$$\langle x - y_0, f(x) \rangle \geq \frac{1}{2} \|y_0 - x\|^2 > 0.$$

Therefore there exists  $\rho > 0$  such that for any  $x \in K$  with  $\|x\| > \rho_0$  there exists  $y$  ( $y = y_0$ ) with  $\|y\| < \|x\|$  and  $\langle x - y, f(x) \rangle \geq 0$ , that is, condition  $(\theta)$  is satisfied. The theorem is now a consequence of Corollary 9, because  $f$  is without exceptional family of elements with respect to  $K$ .  $\square$

By the next definition we will introduce a generalization of a condition, known in Complementarity Theory under the name *Harker-Pang condition*.

**DEFINITION 4.** *We say that a mapping  $f : H \rightarrow H$  satisfies condition (HP) with respect to  $K$ , if there exists a non-empty bounded set  $D \subset K$ , such that the set  $K(D) = \{x \in K \mid \langle f(x), y - x \rangle > 0 \text{ for all } y \in D\}$  is bounded or empty.*

We have the following result.

**THEOREM 11.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $f : H \rightarrow H$  a completely continuous field, or an  $\alpha$ -condensing field or a projectionally pseudo-contractant field. If  $f$  satisfies condition (HP), then the problem  $NCP(f, K)$  has a solution.*

*Proof.* The set  $D \cup K(D)$  is bounded. Therefore there exists  $\rho > 0$  such that for all  $x \in D \cup K(D)$  we have that  $\|x\| \leq \rho$ . If  $x \in K$  is such that  $\rho < \|x\|$ , then for some  $y \in D$  we have  $\langle f(x), y - x \rangle \leq 0$  or  $\langle f(x), x - y \rangle \geq 0$ .

Because  $\|y\| \leq \rho < \|x\|$  we have that  $f$  satisfies condition  $(\theta)$  with respect to  $K$ . The theorem is a consequence of Corollary 9 because  $f$  is without exceptional family of elements with respect to  $K$ .  $\square$

**REMARK.** When  $H = R^n$  is endowed with the Euclidean structure and  $D$  is a singleton set we have that the condition (HP) is the Harker-Pang condition.

Now we consider a general variational inequality. Let  $T : H \rightarrow H$  be a set-valued mapping. We say that  $T$  is a *completely upper semicontinuous* (c.u.s.c.) if  $T$  is upper

semicontinuous and for any bounded set  $B \subset H$ , we have that  $T(B) = \bigcup_{x \in B} T(x)$  is a relatively compact set.

If a set-valued mapping  $f : H \rightarrow H$  has a representation of the form  $f(x) = x - T(x)$ , for all  $x \in H$ , where  $T : H \rightarrow H$  is a c.u.s.c. set-valued mapping, then in this case we say that  $f$  is a c.u.s.c. set-valued field.

**DEFINITION 5.** *We say that  $\{x_r\}_{r>0} \subset H$  is an exceptional family of elements for c.u.s.c. set-valued field  $f : H \rightarrow H$  (where  $f(x) = x - T(x)$ ), with respect to a closed convex subset  $\Omega \subset H$  if the following conditions are satisfied*

- (1)  $\|x_r\| \rightarrow +\infty$  as  $r \rightarrow +\infty$ ,
- (2) for any  $r > 0$  there exists a real number  $\mu_r > 1$  and an element  $y_r \in T(x_r)$  such that  $\mu_r x_r \in \Omega$  and  $y_r - \mu_r x_r \in N_\Omega(\mu_r x_r)$ .

**THEOREM 12.** [Leray-Schauder Type Alternative] *Let  $X$  be a closed subset of a locally convex space  $E$  such that  $0 \in \text{int}(X)$  and  $f : X \rightarrow E$  a compact upper semicontinuous set-valued mapping with non-empty compact contractible values. If  $f$  is fixed-point free, then it satisfies the following condition: there exists  $(\lambda_*, x_*) \in ]0, 1[ \times \partial X$  such that  $x_* \in \lambda_* f(x_*)$ .*

*Proof.* This result is a part of Corollary of the main theorem proved in [7]. □

We have the following alternative theorem for multi-valued variational inequalities.

**THEOREM 13.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $\Omega \subset H$  an arbitrary unbounded closed convex set and  $f(x) = x - T(x)$  a c.u.s.c. set-valued field defined on  $H$  such that  $T$  is with non-empty compact contractible values. Then the problem  $MVI(f, \Omega)$  has at least one of the following properties:*

- (1)  $MVI(f, \Omega)$  has a solution,
- (2) the c.u.s.c. set-valued field  $f$  has an exceptional family of elements with respect to  $\Omega$ .

*Proof.* It is known that the problem  $MVI(f, K)$  has a solution if and only if the set-valued mapping

$$\phi(x) = P_\Omega[x - f(x)] = P_\Omega[T(x)], \quad x \in H$$

has a fixed point (in  $H$ ).

If the problem  $MVI(f, \Omega)$  has a solution, then the proof is finished.

Suppose the problem  $MVI(f, \Omega)$  is without solution. For any positive real number  $r$  (i.e.  $r > 0$ ) we consider the set  $B_r = \{x \in H \mid \|x\| \leq r\}$ .

Obviously  $0 \in \text{int } B_r$ . The set-valued mapping  $\phi$  is fixed-point free with respect to any set  $B_r$  ( $r > 0$ ). Observe that all the assumptions of Theorem 12 are satisfied, since we can show (see the proof of Theorem 7) that the set-valued mapping  $\phi$  is with non-empty compact contractible values. Certainly  $\phi|_{B_r}$  is a compact upper semicontinuous set-valued mapping. Therefore, for any  $r > 0$  there exists an element  $x_r \in \partial B_r$  such that  $x_r \in \lambda_r P_\Omega(T(x_r))$ , for some  $\lambda_r \in ]0, 1[$ .

Hence, there exist  $x_r \in \partial B_r$ ,  $\lambda_r \in ]0, 1[$  and  $y_r \in T(x_r)$  such that  $x_r = \lambda_r P_\Omega(y_r)$ .

By Proposition 1 we have that  $y_r \in \frac{1}{\lambda_r}x_r + N_\Omega(\frac{1}{\lambda_r}x_r)$ . If we denote by  $\mu_r = \frac{1}{\lambda_r}$ , for any  $r > 0$ , then we have

- (1)  $\|x_r\| = r$  and  $\mu_r > 1$  for all  $r > 0$ ,
- (2)  $\mu_r x_r \in \Omega$ , for all  $r > 0$  and
- (3)  $y_r - \mu_r x_r \in N_\Omega(\mu_r x_r)$ ,

that is  $\{x_r\}_{r>0}$  is an exceptional family of elements for  $f$ , with respect to  $\Omega$  and the proof is complete.  $\square$

**COROLLARY 14.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $\Omega \subset H$  an arbitrary unbounded closed convex set and  $f(x) = x - T(x)$  an c.u.s.c. set-valued field defined on  $H$ . If  $T$  is with non-empty compact contractible values and  $f$  is without an exceptional family of elements with respect to  $\Omega$ , then the problem  $MVI(f, \Omega)$  has a solution.*

In our paper [46] are presented several classes of mappings without exceptional family of elements, with respect to a closed convex set.

If we extend these classes to set-valued mappings, we obtain several examples of set-valued mappings without exceptional family of elements for which the problem  $MVI(f, \Omega)$  is solvable.

## 6. COMPLEMENTARITY PROBLEMS WITH PSEUDO-MONOTONE OPERATORS

Many authors studied monotone and pseudo-monotone operators in relations with variational inequalities and complementarity problems [8]-[10];[17]-[20]; [31]-[33]; [50]-[52], [64]-[65].

In this section we will introduce a generalization of pseudo-monotonicity. We will prove a transitivity principle, which implies that the problem  $NCP(f, K)$ , where  $f$  is a pseudo-monotone operator, has a solution if and only if  $f$  is without exceptional family of elements, with respect to  $K$ .

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $K \subset H$  a closed pointed convex cone and  $f, g : H \rightarrow H$  two mappings.

**DEFINITION 6.** *We say that  $f$  is asymptotically  $g$ -pseudo-monotone, with respect to  $K$ , if there exists a real number  $\rho > 0$  such that for all  $x, y \in K$ , with  $\max\{\rho, \|y\|\} < \|x\|$  we have that  $\langle x - y, g(y) \rangle \geq 0$  implies  $\langle x - y, f(x) \rangle \geq 0$ .*

**REMARK.** If  $f$  is pseudo-monotone, then it is asymptotically  $f$ -pseudo-monotone.

We have the following result.

**THEOREM 15.** [Transitivity Principle] *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $K \subset H$  a closed pointed convex cone and  $f, g : H \rightarrow H$  two mappings. If the following assumptions are satisfied*

- (1)  *$f$  is a completely continuous field or an  $\alpha$ -condensing field,*
- (2)  *$f$  is asymptotically  $g$ -pseudo-monotone,*
- (3) *the problem  $NCP(g, K)$  has a solution,*

*then the mapping  $f$  is without exceptional family of elements and the problem  $NCP(f, K)$  has a solution.*

*Proof.* Let  $x_* \in K$  be a solution of the problem  $NCP(g, K)$ . By the Complementarity Theory we know that  $x_*$  is a solution of the following variational inequality

$$(1) \quad \langle x - x_*, y(x_*) \rangle \geq 0, \text{ for all } x \in K.$$

Since  $f$  is asymptotically  $g$ -pseudo-monotone we have that (1) implies

$$(2) \quad \begin{cases} \langle x - x_*, f(x) \rangle \geq 0, \text{ for all } x \in K \\ \text{with } \max\{\rho, \|x_*\|\} < \|x\| \end{cases}$$

Suppose that  $f$  has an exceptional family of elements, that is there exists a family of elements  $\{x_r\}_{r>0} \subset K$  such that for any  $r > 0$  there exists  $\mu_r > 0$  with the following properties:

- (i)  $u_r = \mu_r x_r + f(x_r) \in K^*$ ,
- (ii)  $\langle x_r, u_r \rangle = 0$ ,
- (iii)  $\|x_r\| \rightarrow \infty$  as  $r \rightarrow \infty$ .

By property (iii) we can choose  $x_r$  such that  $\max\{\rho, \|x_*\|\} < \|x_r\|$ . Making use of properties (i), (ii) and considering formula (2) we obtain

$$\begin{aligned} 0 &\leq \langle x_r - x_*, f(x_r) \rangle = \langle x_r - x_*, u_r - \mu_r x_r \rangle \\ &= \langle x_r, u_r \rangle - \langle x_*, u_r \rangle - \mu_r \|x_r\|^2 + \mu_r \langle x_*, x_r \rangle \\ &\leq -\mu_r \|x_r\|^2 + \mu_r \|x_*\| \|x_r\| = \mu_r \|x_r\| [\|x_*\| - \|x_r\|] < 0, \end{aligned}$$

which is a contradiction.

Therefore,  $f$  is without exceptional family of elements with respect to  $K$ . By Corollary 9 (of Theorem 8) we have that the problem  $NCP(f, K)$  has a solution and the proof is complete.  $\square$

**COROLLARY 16.** *Let  $f, g : R^n \rightarrow R^n$  be continuous mappings and  $K \subset R^n$  an arbitrary closed pointed convex cone. If the following assumptions are satisfied*

- (1)  *$f$  is asymptotically  $g$ -pseudo-monotone*
- (2) *the problem  $NCP(g, K)$  has a solution,*

*then the problem  $NCP(f, K)$  has a solution.*

The next corollary is a result obtained recently by G. Isac and V. Kalashnikov [48].

**COROLLARY 17.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $K \subset H$  a closed pointed convex cone and  $f : H \rightarrow H$  a completely continuous field or an  $\alpha$ -condensing field. If  $f$  is pseudo-monotone then the problem  $NCP(f, K)$  has a solution if and only if  $f$  is without exceptional family of elements with respect to  $K$ .*

**REMARK.** Theorem 15 may have interesting applications to the study of economic equilibrium of two economic systems when one is dependent on the other.

## 7. LOCAL PROJECTED DYNAMICAL SYSTEMS, VARIATIONAL INEQUALITIES AND DIFFERENTIAL INCLUSIONS

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $D \subset H$  a closed convex set. Let  $P_D$  be the projection operator onto  $D$ .

**THEOREM 18.** *If  $D \subset H$  is a closed convex set, then  $P_D$  is directionally differentiable at every point  $x_0 \in D$ .*

*Proof.* For a proof of this result the reader is referred to [78] and particularly to [68].  $\square$

Consequently, given  $x \in D$  and a continuous mapping  $f : D \rightarrow H$ , we have that

$$\lim_{\delta \rightarrow 0^+} \frac{P_D(x - \delta f(x)) - x}{\delta}$$

is well defined. We denote by

$$\Pi_D(x, -f(x)) := \lim_{\delta \rightarrow 0^+} \frac{P_D(x - \delta f(x)) - x}{\delta}.$$

It is known that  $\Pi_D(x, -f(x)) = P_{T_D(x)}(-f(x))$  (see [2]).

Consider the ordinary differential equation

$$(3) \quad \frac{dx}{dt} = \Pi_D(x, -f(x)), \text{ where } x : [0, +\infty[ \rightarrow H.$$

**REMARK.** The right-hand side of this differential equation is discontinuous on the boundary  $\partial D$  of  $D$  and hence, the study of this equation is not easy.

Equation (3) was considered and used systematically in  $R^n$  in the study of local projected dynamical systems associated to some problems in Economics [63] and [13],[14],[21],[23],[34],[35], [57]-[63] and [72].

Now, in this paper, we consider equation (3) in an arbitrary Hilbert space and we are interested, in particular, in the study of the stability of solutions of this equation.

We say that a function  $x : [0, \infty[ \rightarrow D$  is a solution to equation (3) if  $x(\cdot)$  is absolutely continuous and if

$$\frac{dx(t)}{dt} = \Pi_D(x(t), -f(x(t)))$$

for all  $t$  save on a set of Lebesgue measure 0.

**DEFINITION 7.** *The local projected dynamical system defined by the mapping  $f$  on the set  $D$  (denoted by  $LPDS(-f, D)$ ) is defined as the mapping  $\Phi : D \times R \rightarrow D$ , where  $\Phi(x, t) = \Phi_x(t)$  solves the initial value problem associated to equation (3), that is,*

$$(4) \quad \frac{d\Phi_x(t)}{dt} = \Pi_D(\Phi_x(t), -f(\Phi_x(t))), \quad \Phi_x(0) = x_0 \in D,$$

save a set of Lebesgue measure 0.

**REMARK.** In the book [63], the system  $LPDS$  is denoted only by  $PDS(f, K)$  and it is named *projected dynamical system*. We use here the term "local" since there exists now in literature the notion of "global projected dynamical system".

Recently we proved the following result, which is a generalization to an arbitrary Hilbert space of Dupuis' Representation Lemma [22],[23],[63], known in  $R^n$ .

Dupuis' Lemma is used in [63] in the study of stability of solutions to the local projected dynamical systems. We denote by  $S = \{x \in H \mid \|x\| = 1\}$  and by  $\bar{S}^\sigma$  the weak closure of  $S$ .

**THEOREM 19.** [Dupuis' Lemma for Hilbert spaces] *If  $\Pi_D(x, v) = \lim_{\delta \rightarrow 0^+} \frac{P_D(x + \delta v) - x}{\delta}$ , then we have*

- (1)  $\Pi_D(x, v) = v$ , if  $x \in \text{int}D$ ,  $\text{int}D \neq \emptyset$ ,
- (2)  $\Pi_D(x, v) = v$ , if  $x \in \partial D$ ,  $\sup_{n \in \bar{S}^\sigma \cap N_D(x)} \langle v, n \rangle < 0$ ,
- (3)  $\Pi_D(x, v) = v - \langle v, n^* \rangle n^*$ , if  $x \in \partial D$ ,  
and  $\sup_{n \in \bar{S}^\sigma \cap N_D(x)} \langle v, n \rangle > 0$ . (In this case  $n^*$  is the unique element  $n$  which achieves the  $\sup_{n \in \bar{S}^\sigma \cap N_D(x)} \langle v, n \rangle$ ),
- (4)  $\Pi_D(x, v) = v$ , if  $x \in \partial D$ ,  $\sup_{n \in \bar{S}^\sigma \cap N_D(x)} \langle v, n \rangle = 0$  and  $\sup_{n \in \bar{S}^\sigma \cap N_D(x)} \langle v, n \rangle$  has at least one maximizer  $n^0 \neq 0$ ,
- (5)  $\Pi_D(x, v) = v - \zeta$ , where  $\zeta \in N_D(x)$  is an undetermined element, if  $x \in \partial D$ ,  
 $\sup_{n \in \bar{S}^\sigma \cap N_D(x)} \langle v, n \rangle = 0$  and  $\sup_{n \in \bar{S}^\sigma \cap N_D(x)} \langle v, n \rangle$  has as maximizer only  $n^* = 0$ .

*Proof.* The proof is more complicated than in  $R^n$  and it is based on several technical intermediate results.  $\square$

The proof of Theorem 19 will be published in another paper.

From Theorem 19 we deduce the following two important consequences.

**COROLLARY 20.** *An element  $x_* \in D$  is a stationary (equilibrium) point of the local projected dynamical system  $LPDS(f, D)$ , i.e.  $\Pi_D(x, -f(x)) = 0$ , if and only if  $x_*$  is a solution of the variational inequality  $VI(f, D)$ .*

**COROLLARY 21.** *An element  $x_* \in D$  is a stationary (equilibrium) point of the local projected dynamical system  $LPDS(f, D)$ , if and only if  $x_*$  is a stationary point of the differential inclusion*

$$(5) \quad \frac{dx}{dt} \in -f(x) - N_D(x),$$

that is  $0 \in -f(x_*) - N_D(x_*)$ .

Therefore, in a general Hilbert space, we must study the solutions to a variational inequality or a complementarity problem, from the dynamical point of view, using not equation (4), as in the Euclidean space, but studying the viable solutions of the differential inclusion (5).

Next we present some results about the stability of solutions to the variational inequality problem  $VI(f, D)$ . We assume  $f$  to be a, respectively, pseudo-monotone, quasi-monotone or strongly pseudo-monotone mapping on  $D$ . We show that each assumption will impose a certain dynamics upon the solution to the  $VI(f, D)$  and implicitly, when  $D$  is a closed convex cone, for solutions to the  $CP(f, K)$ .

We say that a solution  $x(t)$  of the differential inclusion (5) is *viable* if this solution starts at the point  $x_0 \in D$  and  $x(t) \in D$  for any  $t$ .

**ASSUMPTION.** We assume that the differential inclusion (5) has viable solutions for each given initial point  $x_0 \in D$ .

The question of existence of solutions to the differential inclusion (5) is a hard problem [69] and it will be treated in a separate paper. In  $R^n$  there exist several existence theorems [2], [13], [14], [27], [28], [34], [35].

Denote by  $x(t) := \Phi_x(t)$  a solution of (5) which starts at the point  $x \in D$ .



**DEFINITION 8.** We say that  $x^* \in D$  is a strict stationary point of the differential inclusion (5) if there exists  $n^* \in N_D(x^*)$  such that

- (1)  $0 = -f(x^*) - n^*$  and
- (2)  $\langle y - x^*, n^* \rangle < 0$  for any  $y \in D$ ,  $y \neq x^*$ .

From Theorem 19 we have that if  $\text{int}D$  is non-empty, a strict stationary point can not be in  $\text{int}D$ .

Because of Theorem 19 we have the following result.

**THEOREM 22.** Let  $x^*$  be a stationary point of the inclusion (5) (i.e. a solution of the VI( $f, D$ )) and  $x(t)$  a viable solution of (5). Then the mapping

$$\dot{D}(t) = \frac{d}{dt} \left( \frac{1}{2} \|x(t) - x^*\|^2 \right)$$

is non-positive, whenever  $f$  is pseudo-monotone, strongly pseudo-monotone or quasi-monotone and  $x^*$  is a strict stationary point.

*Proof.* We have that

$$(6) \quad \dot{D}(t) = \frac{d}{dt} \left( \frac{1}{2} \|x(t) - x^*\|^2 \right) = \langle x(t) - x^*, \dot{x}(t) \rangle.$$

If  $x(t)$  is a solution to the differential inclusion (5), then  $\frac{dx(t)}{dt} \in -f(x(t)) - N_D(x(t))$ , which implies there exists  $n \in N_D(x(t))$  such that  $\frac{dx(t)}{dt} = -f(x(t)) - n$ . Then (6) becomes

$$\dot{D}(t) = \langle x(t) - x^*, -f(x(t)) - n \rangle = \langle x(t) - x^*, -f(x(t)) \rangle + \langle x(t) - x^*, -n \rangle.$$

Now  $\langle x(t) - x^*, -n \rangle = \langle x^* - x(t), n \rangle \leq 0$  from the definition of  $N_D(x(t))$ . The expression  $\langle x(t) - x^*, -f(x(t)) \rangle$  is non-positive if  $\langle x(t) - x^*, -f(x^*) \rangle$  is non-positive and this implication is true if  $f$  is pseudo-monotone, strongly pseudo-monotone or quasi-monotone and  $x^*$  is a strict stationary point.

Since  $x^*$  is a stationary point of (5), then  $0 \in -f(x^*) - N_D(x^*)$ , in other words, there exists a  $n^* \in N_D(x^*)$  such that  $-f(x^*) - n^* = 0$ , i.e.  $-f(x^*) = n^*$ .

This implies that

$$\langle x(t) - x^*, -f(x^*) \rangle = \langle x(t) - x^*, n^* \rangle \leq 0,$$

from the definition of  $N_D(x^*)$ .

Therefore, when  $f$  is pseudo-monotone, strongly pseudo-monotone or quasi-monotone and  $x^*$  is a strict stationary point we obtain

$$\dot{D}(t) = \frac{d}{dt} \left( \frac{1}{2} \|x(t) - x^*\|^2 \right) \leq 0$$

and the proof is complete. □

We introduce the following definition.

**DEFINITION 9.** A stationary point  $x^*$  of (5) is a monotone attractor if  $\exists \delta > 0$  such that,  $\forall x \in B(x^*, \delta)$ , the function  $d(x, t) := \|x(t) - x^*\|$  is non-increasing as function of  $t \geq 0$ , where  $x(t)$  is a viable solution of (5), starting at the point  $x$ .

**REMARK.** The notion of a monotone attractor, introduced in Definition 9, is different than the one of attractor from the classical theory of dynamical systems.

**THEOREM 23.** *Let  $x^*$  be a stationary point of (5). If  $f$  is pseudo-monotone then  $x^*$  is a monotone attractor. We have the same result if  $f$  is quasi-monotone and  $x^*$  is a strict stationary point.*

*Proof.* We show that  $d(x, t) := \|x(t) - x^*\|$  is a non-increasing function of  $t$ .

Indeed, if we suppose that

$$t_1 < t_2 \implies \|x(t_1) - x^*\| < \|x(t_2) - x^*\|,$$

then we have

$$\|x(t_1) - x^*\|^2 < \|x(t_2) - x^*\|^2,$$

which is impossible because of Theorem 22. Therefore  $x^*$  is a monotone attractor and the proof is complete.  $\square$

In the book [63], several stability results are proved in the Euclidean space  $(R^n, \langle \cdot, \cdot \rangle)$ , using the non-positivity of the function  $\dot{D}(t)$  considered in Theorem 22. From Theorem 22 we have now that several of these stability results are valid in an arbitrary Hilbert space for viable solutions of differential inclusion (5).

From the theory of differential equations in Banach spaces we know that we can find more easily *weak solutions*, if we consider the space  $H$  with its weak topology. Because of this fact we introduce the following notion.

Given a function  $x : [0, \infty[ \rightarrow H$ , we say that an element  $[x'(t_0)]_w \in H$  is the *weak derivative* of  $x$  at the point  $t_0 \in [0, \infty[$  if and only if

$$\lim_{t \rightarrow t_0} \frac{1}{t - t_0} \langle \zeta, x(t) - x(t_0) \rangle = \langle \zeta, [x'(t_0)]_w \rangle$$

for any  $\zeta \in H$ .

**DEFINITION 10.** *We say that a function  $x : [0, \infty[ \rightarrow H$  is a weak viable solution of the differential inclusion (5) if*

$$\begin{cases} [\frac{dx(t)}{dt}]_w \in -f(x(t)) - N_D(x(t)) \\ x(t_0) = x_0 \in D \end{cases}$$

and  $x(t) \in D$  for any  $t \geq 0$ .

Recently, we obtained the following result.

**THEOREM 24.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $x : [0, \infty[ \rightarrow H$ ,  $t_0 \geq 0$ . If  $[x'(t_0)]_w$  is the weak derivative of  $x$  at the point  $t_0$ , then we have*

$$\lim_{t \rightarrow t_0} \frac{1}{t - t_0} \left( \frac{\|x(t)\|^2}{2} - \frac{\|x(t_0)\|^2}{2} \right) = \langle x(t_0), [x'(t_0)]_w \rangle.$$

*Proof.* The proof of this result will be published in a recent paper.  $\square$

From Theorem 24 we obtain the following result.

**THEOREM 25.** *Let  $x^*$  be a stationary point of the inclusion (5) (i.e. a solution of the VI( $f, D$ )) and  $x(t)$  a viable weak solution of (5), starting at the point  $x$ . Then the mapping*

$$\dot{D}(t) = \frac{d}{dt} \left( \frac{1}{2} \|x(t) - x^*\|^2 \right)$$

*is non-positive, whenever  $f$  is pseudo-monotone, strongly pseudo-monotone or quasi-monotone and  $x^*$  is a strict stationary point.*

*Proof.* The proof is similar to the proof of Theorem 22 and it is also based on Theorem 24. □

For weak solutions, Theorem 23 has the following form.

**THEOREM 26.** *Let  $x^*$  be a stationary point of (5). If  $f$  is pseudo-monotone then  $x^*$  is a monotone attractor for weak viable solutions of (5). We have the same result if  $f$  is quasi-monotone and  $x^*$  is a strict stationary point.*

## 8. COMMENTS AND OPEN PROBLEMS

The results presented in this paper, related to the study of stability of solutions to variational inequalities or to complementarity problems in infinite dimensional Hilbert spaces, must be based on the study of viable solutions to differential inclusion (5). It seems also to be interesting to study the weak viable solutions of differential inclusion (5). If we replace  $N_D(x)$  in (5) by  $N_D(x) \cap \bar{B}(0, 1)$  (which is weakly compact) then does the set-valued mapping  $x \rightarrow -f(x) - N_D(x) \cap \bar{B}(0, 1)$  have a weakly continuous selection?

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