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ON THE SIGNIFICANCE OF INTEGRAL PROPERTIES OF ORBITS IN SOME SUPERLINEAR FIXED-PERIOD PROBLEMS

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Abstract. Consider the nonlinear two-point boundary value problem

$$\begin{aligned} &u_{xx} + u^p = 0, \\ &u(0) = u(L) = 0, \\ &p > 1, \quad L > 0. \end{aligned}$$

For every $p = 2j - 1$, $j \in \mathbb{N}$, $j \geq 2$ the corresponding positive solution has a constant moment, i.e. it does not depend on L. First, we justify these integral properties in the framework of Hamiltonian mechanics.

Then, making use of a shooting method, we transform the above problem into an initial value one for a system of differential equations. The integral properties mean the existence of a Hamiltonian (non-quadratic algebraic invariant) for this system. Two numerical methods (a 2 stage Runge-Kutta and a Stormer-Verlet), which preserve the Hamiltonian, are used to integrate the system.

Keywords: Hamiltonian systems, positive solutions, constant moments, symplectic integrators

AMS Subject Classification: 34A34, 34B15, 34C25, 37K05, 65P10

1. Introduction

Given a bounded interval $[0, L]$, $L > 0$ let us consider the nonlinear two-point boundary value problem:

$$\begin{cases} u_{xx} + u^p = 0, \ 0 < x < L, p = 2j - 1, j \in \mathbb{N}, j \geq 2 \\ u(0) = u(L) = 0. \end{cases} \tag{1}$$

First of all we show that for every odd $p, p > 1$, the moments of the corresponding (unique) positive solution of (1) are constants; i.e. they do not depend on L. More explicitly we show that

$$M_{j-1} = \pi / \sqrt{j}, \tag{2}$$

where, we adopt the notation $M_n := \int\limits_0^L u^n(x)\, dx$ for the n th moment of u. At this point we give an alternative proof for the solution published in [12] (we are among the solvers) and refine some results from [4]. We believe that is fairly interesting to understand the sense of these properties from numerical point of view. Thus, in our previous paper [7], we developed an *indirect algorithm* (in the spirit exposed in [13]) to find the positive solution to (1). That was a projection - like algorithm for the following initial - boundary value parabolic problem attached to (1):

$$\begin{array}{lll} & u_t = u_{xx} + u^p, & 0 < x < L,\ o < t < T \\ (3) & u(0,t) = u(L,t) = 0, & 0 < t < T \\ & u(x,0) = u_0(x) & 0 < x < L \end{array}$$

where $u_0(0) = u_0(L) = 0$. In this context (2) represented a hyperplane in $L_2(0,L)$ and our algorithm kept every numerical approximation into this hyperplane. Analytically ([3], [10]) as well as numerically ([1], [2] and our numerical experiments reported in [7]) is proved that the superlinear problem (3) belongs to the blow up case. To avoid such behaviour the integral property (2) showed its usefulness. In [6] we developed a *direct algorithm* where the same property was considered as a restriction in a Lagrange's multiplier method.

In this paper, in the framework of Hamiltonian setting ([5]) we use two numerical method that preserve the Hamiltonian of the problem.

2. Integral properties of positive solution to (1)

In our previous paper [6] some particularities of the set of solutions of the autonomous problem (1) have been summarized. Thus, we used an ad-hoc method to show the bifurcation from infinity and gave a variational characterization of the positive solution. Our findings are in accordance with those in [11] and express the fact that there is no finite value of parameter L at which bifurcation occurs and there do exist solutions whose norm tends to zero as the parameter tends to infinity. More than that these small bifurcation solutions are characterized by strongly nonlinear balances.

Thus, to justify (2) in the Hamiltonian setting we rewrite (1) as

$$(4) \qquad (u'(x))^2 = (u'(0))^2 - \frac{1}{j}\left(u^{2j}(x)\right),\ 0 < x < L,$$

or explicitly

$$(5) \qquad u'(x) = \pm\frac{1}{\sqrt{j}}\sqrt{\left(\sqrt{j}u'(0)\right)^2 - u^{2j}(x)}.$$

We also observe that there exists a unique point $c \in (0,1)$ where the first derivative of the positive solution vanishes and consequently u assumes its maximum value on $[0,L]$. Let us denote $u_{\max} := u(c)$. For this maximum value we have from the first integral (4)

$$(6) \qquad u_{\max}^j = \sqrt{j}u'(0).$$

From (5), (6) and the monotonicity of u we deduce

$$\int_0^L u^{j-1}dx = \frac{2\sqrt{j}}{j}\int_0^{u_{\max}} \frac{du^j}{\sqrt{\left(\sqrt{j}u'(0)\right)^2 - u^{2j}}} = \frac{2}{\sqrt{j}}\arcsin\frac{u^j}{\sqrt{j}u'(0)}\Bigg|_0^{u_{\max}} = \frac{\pi}{\sqrt{j}}\ ,$$

which means precisely (2).

3. The Hamiltonian setting for (1)

Via a shooting argument the problem (1) can be written as

$$\left\{\begin{array}{c} u_x = v \\ v_x = -u^p \\ u(0) = 0 \\ v(0) = \alpha \end{array}\right. \tag{7}$$

for an undetermined parameter α. The first integral (4) leads to the Hamiltonian

$$H(v,u) = \frac{1}{2}v^2 + \frac{1}{2j}\left(u^{2j}\right) = \frac{1}{2}u'(0) = const. \tag{H}$$

4. Numerical methods that preserve non-quadratic algebraic invariant (H)

Given the differential system (7) in tandem with the invariance condition (H) we have determined a 2 stage Runge-Kutta method of order 4 whose matrix and weights are displayed in Table 1 .

$1/2-\sqrt{3}/6$	$1/4$	$1/4-\sqrt{3}/6$
$1/2+\sqrt{3}/6$	$1/4+\sqrt{3}/6$	$1/4$
	$1/2$	$1/2$

Table 1. The $R-K$matrix and $R-K$weights

This method satisfies the condition (H) with the precision of a term $O\left(h^3\right)$. We have made use of the theory from [9] where a necessary condition for a Runge-Kutta $(R-K)$ method to obey the invariant (H) is obtained. In [14] the same method is obtained using a strategy based on the conservation of symplectic structure. The numerical results carried out using this method are displayed in Fig.1.

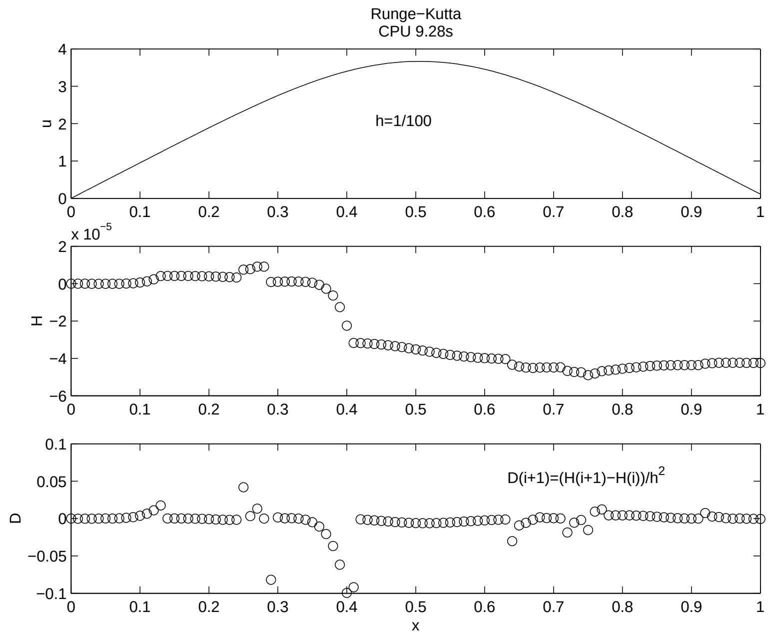
We also have used a Stormer-Verlet $(S-V)$ method ([8]) which, for our problem, reads

$$\left\{\begin{array}{l} u_0 = 0,\ v_0 = 9.5 \\ v_{i+1/2} = v_i - \frac{h}{2}u_i^p \\ u_{i+1} = u_i + hv_{i+1/2} \\ v_{i+1} = v_{i+1/2} - \frac{h}{2}u_{i+1}^p,\ i = 1,2,...,N \\ H_i := v_i^2 + \frac{1}{2}u_i^4 - v_0^2. \end{array}\right.$$

The corresponding numerical results are reported in Fig.2. We have to remark that the value of shooting parameter α was obtained by numerical experiments. We have not paied special heed to this aspect.

5. Concluding remarks and open problems

Our numerical experiments show clearly that the implicit Runge-Kutta method is much more expensive, with respect to the CPU time, than the Storner-Verlet method. Howerer, both methods produce positive solutions to (1) which agree fairly well with our previous results. More than that, the oscillatory variation of Hamiltonian energy, which is typical of symplectic methods, is apparent at least for S-V method. As open problems, we mention first, the lack of sufficient conditions for the retention of non-quadratic conservation laws, and second, the extension of our strategy to multidimensional radially symmetric problems (1).



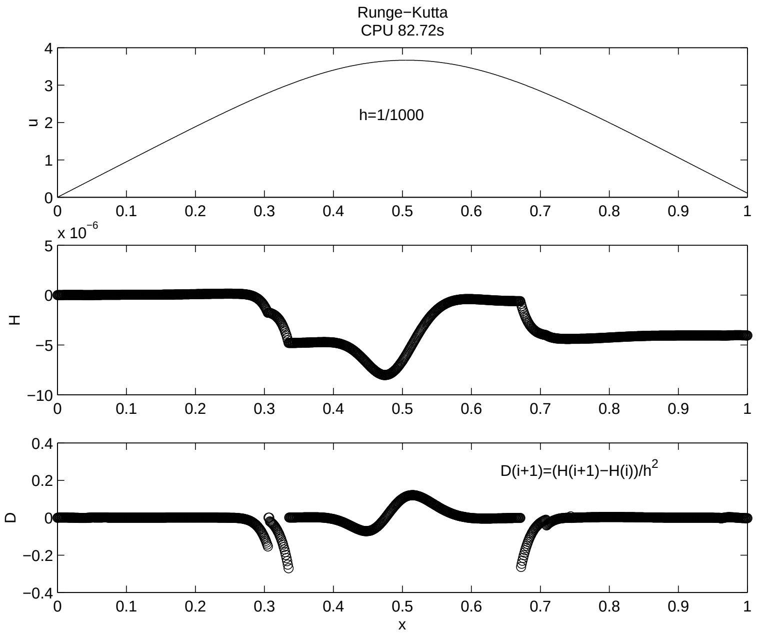


Figure 1. The positive solution to (1), the variation of the Hamiltonian energy and the discrete values of the term $H = 0\left(h^2\right)$corrersponding to $R - K$method a/ h = 1/100, b/ h = 1000

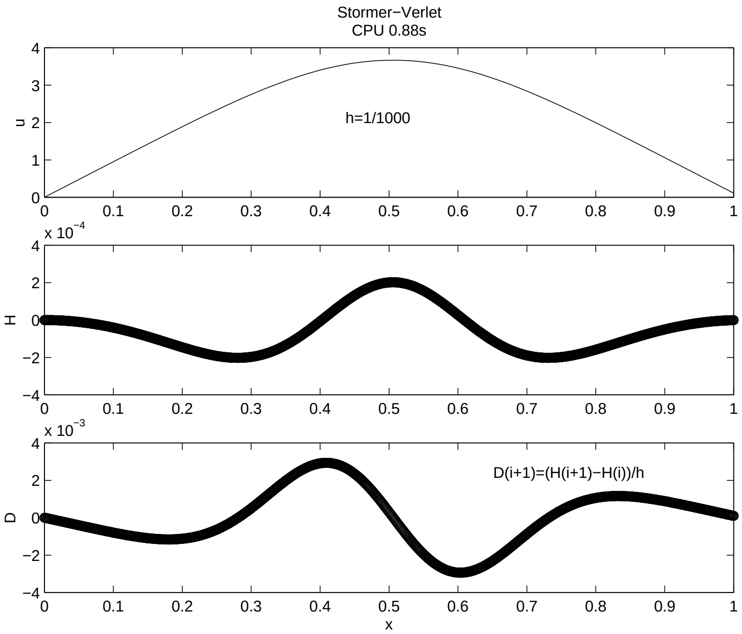


Figure 2. The positive solution to (1), the oscillatory variation of the Hamiltonian energy and the discrete values of the term H = 0(h) corresponding to S-V method.

References

[1] Budd, C.J., Dold, J. W. and Stuart, A. M., Blow-up in a System of Partial Differential Equations with Conserved First Integral, SIAM J. Appl. Math. 53(1993), 718-742.

[2] Budd, C. J., Dold, J. W. and Stuart, A. M., Blow-up in a System of Partial Differential Equations with Conserved first Integral. Part. II: Problems with Convection, SIAM J. Appl. Math. 54(1994), 610-640.

[3] Deng, K., Levine, H. A., The Role of Critical Exponents in Blow-up Theorems: the sequel, J. Math. Anal. Appl. 243(2000), 85-126.

[4] Dold, J. W., A Note of Integral Properties of Periodic Orbits, SIAM Reviews, 35(1993), 125-129.

[5] Ekeland, I., Convexity Methods in Hamiltoniam Mechanics, Springer-Verlag, 1990.

[6] Gheorghiu, C. I., Trif, D., On the Bifurcation and Variational Approximation of the Positive Solution of a Nonlinear Reaction-Diffusion Problem, Studia Univ. "Babeş-Bolyai", Mathematica, XLV (2000), 29-37.

[7] Gheorghiu, C. I., Trif, D., The Numerical Approximation to Positive Solution for Some Reaction-Diffusion Problems, PU.M.A. 11(2001), 243-253.

[8] Hairer, E., Numerical Geometric Integration, Univ. Genève, March 1999.

[9] Iserles, A., Zanna, A., Preserving Algebraic Invariants with Runge-Kutta Methods, Elsevier Preprint, sept. 1999.

[10] Levine, A. H., The Role of Critical Exponents in Blow-up Theorems, SIAM Reviews 32(1990), 262-288.

[11] Rosenblat, S., Davis, S. H., Bifurcation from Infinity, SIAM J. Appl. Math. 37(1979), 1-19,

[12] Rulla, J. et al., Average Temperature in a Reaction-diffusion Process, SIAM Reviews 40(1998), 265-291.

[13] Stuart, A. M., Humphries, A. R., Dynamical Systems and Numerical Analysis, Cambridge Univ. Press, 1996.

[14] Yoshida, H., Recent progress in the theory and applications of symplectic integrators, Celestial Mechanics and Dynamical Astronomy 56(1993), 27-43