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FIXED POINTS, DIFFERENTIAL EQUATIONS, AND PROPER MAPPINGS

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Abstract. This is an expository paper in which we investigate the problem of finding the proper mapping to obtain the solution of a differential equation. The classical existence theorem for differential equations begins by telling us to write the differential equation as an integral equation; and we soon see good reasons for doing so. We ask if there are even better reasons for not doing this. During the last several years we have studied the problem of proving existence and periodic results by mapping the differential equation itself without converting to an integral equation. We have studied the problem of proving stability results by converting to an integral equation and then applying fixed point theory. Our main quest here is to discover how to use fixed point theory to prove stability without writing the differential equation as an integral equation. We leave this mainly as an open problem.
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0. Stating the problem

The investigator begins with a differential equation DE and the need to find a solution ϕ with a given set of properties and hopes to arrive at the solution by means of a fixed point theorem. The first step is to discover the space S in which ϕ should reside. The next step is to discover a mapping $P : S \to S$ so that if P has a fixed point ψ, then ψ will qualify as a solution of DE. The third step is to find a fixed point theorem which is compatable with the previous two steps. Every step is often a great challenge.

For example, when investigators wanted to prove that a functional differential equation with infinite delay had a periodic solution, it was the greatest surprise to find that compactness requirements forced them to work in a space with unbounded initial functions [5]. Moreover, we learn so much in the process. Early in our studies we find the theorem which says that a continuous function on a compact set is uniformly continuous. Thus, we naturally think that compactness is a great friend of continuity. It is in fixed point theory that we really see that continuity and compactness are constantly at war with one another. We must weaken a space in order to make a set compact, but this means that we must strengthen the properties of the mapping so that it will still be continuous.

Often there is a very simple trick that enables the investigator to arrive at a suitable mapping. Consider the implicit function problem of finding a function ϕ satisfying the equation

$$f(t, y) = 0.$$

We have seen it solved so often by the contraction mapping principle that we seldom think of the creativity involved in writing down

$$(P\phi)(t) = \phi(t) + f(t, \phi(t))$$

so that if $P\phi = \phi$ then the extraneous part of the equation drops out and we have $f(t, \phi(t)) = 0$.

In this expository paper we describe some of the problems encountered in trying to prove existence, periodicity, and stability.

We begin with existence and note that the traditional manner of obtaining the mapping is to integrate the equation; this works well until we come to neutral functional differential equations. We then find that integrating the equation does not work, but it does work to integrate the derivative of the solution. This gives rise to what we call direct fixed point mappings. Two examples are given of classical problems from the stated point of view, but the reader is referred to another paper for the neutral case.

If we want to prove that a differential equation has a periodic solution and if we know, for example, that solutions are uniformly ultimately bounded, then a Poincaré map may very effectively work with a sandwich fixed point theorem [11] to solve the problem. But if little is known about solutions, then the investigator may try to define a mapping by integrating the equation. This process is filled with difficulties. We do not know the initial condition so we do not know the lower limit of integration; but even if we knew the lower limit, we frequently obtain a mapping equation which maps periodic functions out of the set. If the equation has a proper linear part then we may use the variation of parameters formula and perform the integration from $-\infty$ to t, circumventing both of the aforementioned difficulties. But if the equation has no linear part then the investigator improvises, perhaps by adding and subtracting a linear term. This, too, can introduce other problems. Once more, we find that by integrating the derivative of the solution instead of integrating the differential equation we will obtain a proper mapping. Again, direct fixed point mappings seem quite successful. An example of an infinite delay problem is given.

For more than a hundred years stability problems have been investigated mainly by Liapunov's direct method. In that theory there arise problems which have proved to be very difficult. One of the major dfficulties involves what is called the annulus argument. The solution may race back and forth across an annulus so rapidly that integration of the derivative of the Liapunov function does not yield a quantity large enough to send the Liapunov function, and hence the solution, to zero. In order to solve such problems we began the study of stability by fixed point theory. The central difficulty is in obtaining a proper mapping. If there is a proper linear part, then we can use the variation of parameters formula and obtain a very effective mapping which eliminates many of the difficultes encountered in Liapunov's direct method. We can also contrive a variation of paramers formula when there is no proper linear part; an example of each is given here. But for a general theory it seems that we really do need to develop a method of direct fixed point mappings for stability problems. In the final section we begin a brief outline of such a process. It is then left as an open problem.

1. Local existence

When we wish to use a fixed point theorem to solve a differential equation we almost always invert the equation and use the inverted form for a mapping. Krasnoselskii studied a paper of Schauder on partial differential equations and concluded that frequently when we invert a perturbed differential operator we obtain the sum of a contraction and compact map. This is so simply seen in the case of existence of solutions of an ordinary differential equation. If the functions are locally Lipschitz, then a simple integration yields a contraction mapping on a sufficiently short interval and we use the contraction mapping theorem to obtain local existence of a unique solution of the initial value problem. The central idea is that the integration parlays the Lipschitz constant (which may be large) into a contraction constant smaller than 1. Since it is a simple symbolic integration there is little reason to not be pleased with the process.

If the functions are merely continuous then that same simple symbolic integration yields a compact map and we use Schauder's second theorem to obtain a solution of the initial value problem, possibly not unique. Again, there is little reason to not be pleased with the process. When we encounter neutral functional differential equations [3] the process breaks down and we are led to other methods.

We begin with two brief proofs of classical results illustrating the techniques. These results are taken from [4]. Classical treatment for Theorems 1.1 and 1.2 is found in Smart [18; pp. 4, 43, 44].

Let $x \in R^n$, $a > 0$, $b > 0$, and

$$\Omega = \{(t, x) \mid |t - t_0| \le a,\ |x - x_0| \le b\}. \tag{1.1}$$

Suppose that $f : \Omega \to R^n$ is continuous and consider the initial value problem

$$x' = f(t, x),\ x(t_0) = x_0. \tag{1.2}$$

THEOREM 1.1 (Cauchy-Picard). If f satisfies a Lipschitz condition on Ω with constant K and if $|f(t, x)| \le M$ on Ω, then (1.2) has a unique solution for $|t - t_0| \le \alpha$ where $0 < \alpha < \min[1/K, b/M, a]$. (By using a different metric than the standard one, $1/K$ can be deleted.)

PROOF. Let $(\mathcal{M}, \rho)$ be the complete metric space of continuous functions where

$$\mathcal{M} = \{\phi : [t_0 - \alpha, t_0 + \alpha] \to R^n \mid \phi(t_0) = f(t_0, x_0), \tag{1.3}$$
$$\|\phi\| \le M,\ \phi \text{ continuous}\},$$
$$\rho(\phi, \psi) = \|\phi - \psi\| = \sup_{|t - t_0| \le \alpha} |\phi(t) - \psi(t)|,$$

$|\cdot|$ is a norm on R^n and also denotes absolute value.

Here, our proof diverges from the classical one. For each $\phi \in \mathcal{M}$ define

$$\Phi(t) = x_0 + \int_{t_0}^{t} \phi(s)ds. \tag{1.4}$$

Thus, $\Phi(t_0) = x_0$, $|\Phi(t) - x_0| \le M|t - t_0| \le b$ and so $|f(t, \Phi(t))| \le M$ since $(t, \Phi(t)) \in \Omega$. Then the mapping P defined on $\mathcal{M}$ by

$$(P\phi)(t) = f(t, \Phi(t)), \; |t - t_0| \le \alpha, \tag{1.5}$$

is continuous, $(P\phi)(t_0) = f(t_0, x_0)$, and $|(P\phi)(t)| \le |f(t, \Phi(t))| \le M$. Hence, $P : \mathcal{M} \to \mathcal{M}$ and if $P\phi = \phi$, then Φ is a solution of (2).

Next, P is a contraction since

$$\begin{aligned} |(P\phi)(t) - (P\psi)(t)| &= |f(t, \Phi(t)) - f(t, \Psi(t))| \\ &\le K|\Phi(t) - \Psi(t)| \\ &= K\left|\int_{t_0}^{t} [\phi(s) - \psi(s)]ds\right| \\ &\le K\alpha\|\phi - \psi\| \end{aligned}$$

and $K\alpha < 1$. Hence, there is a unique fixed point. Φ solves (1.2) and, clearly, is unique.

THEOREM 1.2 (Cauchy-Euler). Let Ω be defined in (1.1), $f : \Omega \to R^n$ be continuous, $|f(t, x)| \le M$ on Ω, and $\alpha = \min[a, b/M]$. Then (1.2) has a solution for $|t - t_0| \le \alpha$.

PROOF. Let $\mathcal{M}$, Φ, and P be defined in (1.3), (1.4), and (1.5). The set $\mathcal{M}$ is contained in a Banach space, $P : \mathcal{M} \to \mathcal{M}$, and P is continuous by the uniform continuity of f on Ω.

We now show that $P\mathcal{M}$ is equicontinuous so that Schauder's second theorem (cf. Smart [18; p. 25]) will yield a fixed point. Let $\epsilon > 0$ be given. We must find $\delta > 0$ so that $\phi \in \mathcal{M}$ and $|t_1 - t_2| < \delta$ imply that $|(P\phi)(t_1) - (P\phi)(t_2)| < \epsilon$. Now $|(P\phi)(t_1) - (P\phi)(t_2)| = |f(t_1, \Phi(t_1)) - f(t_2, \Phi(t_2))|$. Since f is uniformly continuous on Ω, for the given $\epsilon > 0$ there is a $\overline{\delta} > 0$ such that $|t_1 - t_2| < \overline{\delta}$ and $|\Phi(t_1) - \Phi(t_2)| < \overline{\delta}$ imply that $|f(t_1, \Phi(t_1)) - f(t_2, \Phi(t_2))| < \epsilon$. But $|\Phi(t_1) - \Phi(t_2)| = \left|\int_{t_1}^{t_2} \phi(s)ds\right| \le M|t_2 - t_1| < \overline{\delta}$ provided that $|t_2 - t_1| < \overline{\delta}/(M + 1) =: \delta$. This completes the proof.

While there seems to be no reason for choosing one type of proof over the other, we showed in [3] that for a strongly nonlinear neutral equation, the latter method worked when the former did not.

2. Periodic Solutions

Suppose that we wish to prove that there is a periodic solution. If we try to simply integrate the equation to obtain a mapping equation, we can not do so because we do not know what the initial condition might be. Moreover, the mapping we construct will not map periodic functions into periodic functions.

If the equation takes the form

$$x' = Ax + f(t, x)$$

where A is a constant $n \times n$ matrix, all of whose characteristic roots have negative real parts, then we employ the variation of parameters formula and integrate from $-\infty$ to t obtaining

$$x(t) = \int_{-\infty}^{t} e^{A(t-s)} f(s, x(s))ds.$$

We then write

$$(P\phi)(t) = \int_{-\infty}^{t} e^{A(t-s)} f(s, \phi(s))ds.$$

We have avoided the initial condition. If f is periodic in the first coordinate, then P maps periodic functions into periodic functions and we are off to a good start in fixed point theory. The mapping smooths nicely.

But if the equation does not have a convenient linear term then we must manufacture a mapping. A time honored process is to add Ax to both sides [7]; and this works sometimes, but it fails more often than it works. Examples of other methods are found in [6] and [15].

Here, we consider an equation which would be effectively destroyed if we added Ax to both sides since the given functions are so small compared to that term. The equation is

$$x'(t) = \int_{-\infty}^{t} C(t, s)g(x(s))ds + p(t) \tag{2.1}$$

where C, g, and p are continuous and

$$C(t + T, s + T) = C(t, s) \quad \text{and} \quad p(t + T) = p(t), \quad \text{some } T > 0, \tag{2.2}$$

with a view to proving that there is a T-periodic solution. When C is of convolution type with compact support then (2.1) can take the form

$$x' = \int_{t-h}^{t} d(t - s)g(x(s))ds + p(t), \tag{2.3}$$

about which much has been written. When $p(t) = 0$ it was used by Volterra [19] to model a population, while Levin and Nohel [16] use it to model a nuclear reactor, and Hale [13; pp. 120–3] points out that it can represent viscoelasticity. Again, when $p(t) = 0$ and when g satisfies some degenerate conditions, then both Levin and Nohel [16] and Hale [13; pp. 120–3] show that it can have periodic solutions. A solution of this problem was given in [4] when g may be large.

For infinite delay equations the sandwich fixed point theorems ([5] and [11]) can be very effective if general properties of solutions are known. But here we work only from the functions given in the differential equations.

The case with g bounded and monotone, $g(0) = 0$, has been of much interest. We first seek conditions to ensure that (2.1) has a T-periodic solution for arbitrary continuous and T-periodic $p(t)$ with

$$\int_{0}^{T} p(s)ds = 0. \tag{2.4}$$

Later, a similar result is obtained for more general g, but small p.

Our technique here was motivated by the fact that there can be enormous difficulties when converting (2.1) to an integral equation, as mentioned earlier. But what is much worse is that in converting (2.1) to an integral equation we are likely to lose a marvelous property of (2.1). If we could use the right-hand-side of (2.1) itself as the mapping, then with g bounded and C absolutely integrable, that mapping would map the whole space into a bounded set. If, in addition, $\partial C/\partial t$ is absolutely integrable then the map would be compact on bounded sets. A whole class of fixed point theorems would then be at our disposal.

Thus, we consider (2.1) with (2.2) and let

$$(\mathcal{P}_T^0, \|\cdot\|) \tag{2.5}$$

be the Banach space of continuous T-periodic functions with the supremum norm and having mean value zero. Note that if x is T-periodic and solves (2.1) then $x' \in P_T^0$. Thus, we define a map P on $\mathcal{P}_T^0$ by $\phi \in \mathcal{P}_T^0$ implies that

$$(P\phi_k)(t) = \int_{-\infty}^{t} C(t,s)g\biggl(k + \int_0^s \phi(u)du\biggr)ds + p(t) \tag{2.6}$$

where k is a constant chosen so that $P\phi_k \in \mathcal{P}_T^0$. Conditions ensuring the existence of such a k are simple, natural, and consistent with traditional assumptions on (2.1).

PROPOSITION 1. Let (2.2) and (2.4) hold, $g* := dg/dx$ be continuous,

$$\int_{-\infty}^{t} |C(t,s)|ds \text{ be bounded,} \tag{2.7}$$

$$g(0) = 0, \quad g*(x) > 0, \tag{2.8}$$

and suppose that C is of one sign and not identically zero. Then there is a unique k so that $P\phi_k$ defined in (2.6) satisfies $P\phi_k \in \mathcal{P}_T^0$.

PROOF. As $g(x) > 0$ if $x > 0$, for any fixed $\phi \in \mathcal{P}_T^0$, there is a $\overline{k} > 0$ with $g\Bigl(\overline{k} + \int_0^t \phi(s)ds\Bigr) > 0$ and, in the same way, a $\overline{\overline{k}}$ with $g\Bigl(-\overline{\overline{k}} + \int_0^t \phi(s)ds\Bigr) < 0$ for all t. Now $\int_0^T \int_{-\infty}^t C(t,s)g\Bigl(k + \int_0^s \phi(u)du\Bigr)ds\,dt + \int_0^T p(s)ds$ is a continuous function of the constant k; moreover, it changes sign and so the required k is assured, yielding $P\phi_k \in \mathcal{P}_T^0$. To see that it is unique, if $k_1 \neq k_2$ and both $P\phi_{k_i} \in \mathcal{P}_T^0$ then for each fixed s in the integrand there is an $\eta(s)$ so that

$$\begin{aligned} 0 &= \int_0^T \int_{-\infty}^t C(t,s)\left[g\biggl(k_1 + \int_0^s \phi(u)du\biggr) - g\biggl(k_2 + \int_0^s \phi(u)du\biggr)\right]ds\,dt \\ &= \int_0^T \int_{-\infty}^t C(t,s)g^*(\eta(s))[k_1 - k_2]ds\,dt \end{aligned}$$

by the mean value theorem for derivatives, where $\eta(t)$ lies between $k_1 + \int_0^t \phi(u)du$ and $k_2 + \int_0^t \phi(u)du$. But $g^* > 0$, $k_1 \neq k_2$, and $C(t,s)$ is of one sign and not identically zero. Hence, the right-hand-side is not zero.

From here on, $P\phi_k = P\phi$.

PROPOSITION 2. Let the conditions of Proposition 1 hold and for each $\phi \in \mathcal{P}_T^0$ pick that unique k and define P by (2.6). Then P is continuous.

PROOF. We will show that if $\phi \in \mathcal{P}_T^0$ is fixed and if $\phi_i \to \phi$, then $P\phi_i \to P\phi$. By way of contradiction, if $P\phi_i \not\to P\phi$, then there is a subsequence, say ϕ_i again, and $\delta > 0$ with $\|P\phi_i - P\phi\| \geq \delta$. As $\phi_i \to \phi$, it is clear that $\int_0^t \phi_i(s)ds \to \int_0^t \phi(s)ds$ so that if k and k_i are the unique constants in the definitions of $P\phi$ and $P\phi_i$, then $k_i \not\to k$. In particular, there is a subsequence, say k_i again, and a $\mu > 0$ with $|k_i - k| \geq \mu$. Thus, for each $s \in [0, T]$ there is an $\eta(s)$ with

$$\begin{aligned} 0 &= \int_0^T \int_{-\infty}^t C(t,s) \left[g\left(k + \int_0^s \phi(u)du\right) - g\left(k_i + \int_0^s \phi_i(u)du\right)\right] ds\, dt \\ &= \int_0^T \int_{-\infty}^t C(t,s) \left[g^*(\eta(s))\left(k - k_i + \int_0^s (\phi(u) - \phi_i(u))du\right)\right] ds\, dt \end{aligned}$$

and this is a contradiction since the right-hand-side is not zero when $\left| \int_0^t (\phi(u) - \phi_i(u))du \right| < \mu/2$. This completes the proof.

THEOREM 2.1. Let the conditions of Prop. 1 hold, let g be bounded, and let

$$\int_{-\infty}^t |\partial C(t,s)\partial t| ds < \infty. \tag{2.9}$$

Then (2.1) has a T-periodic solution.

PROOF. The map P defined by (2.6) maps $\mathcal{P}_T^0$ into a bounded subset of $\mathcal{P}_T^0$ (bound M) and it is continuous. Let $\mathcal{S} = \{\phi \in \mathcal{P}_T^0 \mid \|\phi\| \leq M + 1\}$. Then $\|P\mathcal{S}\| \leq M$. Prop. 1 shows P is well-defined, while Prop. 2 shows P is continuous. Now we show that $P\mathcal{S}$ is equicontinuous. To this end, let $\epsilon > 0$ be given. We will find $\delta > 0$ such that $\phi \in \mathcal{S}$ and $|t_1 - t_2| < \delta$ imply that $|(P\phi)(t_1) - (P\phi)(t_2)| < \epsilon$. As p is continuous, it suffices to show that there is a bound on $[(P\phi)(t)]'$ for $\phi \in \mathcal{S}$. This is clear from boundedness of

$$C(t,t)g\left(k + \int_0^t \phi(s)ds\right) + \int_{-\infty}^t (\partial C(t,s)/\partial t) g\left(k + \int_0^s \phi(u)du\right)ds.$$

The application of Schauder's second theorem (cf. Smart [18; p. 24] now yields the fixed point and completes the proof.

If p is small and if $g^* > 0$ near $x = 0$, then there is a parallel result even when g is not bounded and monotone. It is convenient to make a change of variable so that $T \leq 1$.

THEOREM 2.2. Let (2.2), (2.4), (2.7), and (2.9) hold, $C(t,s)$ have one sign, and $C(t,s)$ not be identically zero. Suppose there are constants M_1, M_2 such that

$$g(0) = 0, \quad g^*(x) > 0 \text{ if } |x| \leq M_1, \tag{2.10}$$

$$|g(x)| \leq M_2 \text{ if } |x| \leq M_1, \tag{2.11}$$

$$\int_{-\infty}^{t} |C(t,s)|M_2 ds + \|p\| < M_1/2 \text{ and } T \le 1. \tag{2.12}$$

Then (2.1) has a T-periodic solution.

PROOF. Let $\mathcal{S} = \{\phi \in \mathcal{P}_T^0 \,\big|\, \|\phi\| < M_1/2\}$ and define P by (2.6) where k is now chosen as follows. If $\phi \in \mathcal{S}$ then there are k_i with $|k_i| \le M_1/2$,

$$-M_1 < k_1 + \int_0^t \phi(s)ds < 0,$$

and

$$0 < k_2 + \int_0^t \phi(s)ds < M_1$$

so that

$$\int_0^T \int_{-\infty}^{t} C(t,s)g\Big(k_i + \int_0^t \phi(u)du\Big)ds\,dt =: J(k_i)$$

changes sign as i changes from 1 to 2. Hence, there is a k with $J(k) = 0$ and $|k| \le M_1/2$. The argument in Proposition 1 will show that k is unique.

Hence, $P : \mathcal{S} \to \mathcal{P}_T^0$ and $\phi \in \mathcal{S}$ implies that

$$\|P\phi\| \le \|p\| + \sup \int_{-\infty}^{t} |C(t,s)|M_2 ds < M_1/2$$

and so $P : \mathcal{S} \to \mathcal{S}$. The remainder of the proof is just like that of Theorem 2.1.

Additional results on the existence of periodic solutions of delay equations using the direct mapping can be found in [12].

3. A Simple Stability Result

We have noted that there is little innovation needed in obtaining existence theory by direct fixed point maps and we have shown how to construct direct fixed point maps for periodic solutions. Now we turn to stability theory and cite two results found in [8]. The first shows a stability result based on a mapping obtained from a linear term and the variation of parameters formula. The second example (in Section 4) concerns a stability result obtained from the variation of parameters formula when there is no linear term and none is introduced. We leave the reader with the problem of how to obtain a direct fixed point mapping parallel to those of our existence or periodic results.

Fixed point theory has been used for a very long time in proving existence, uniqueness, and periodicity of solutions of ordinary and functional differential equations. But we believe that such use in the study of stability is fairly new. In [8] we give an extensive discussion motivating such a study.

The half-linear equation

$$x'(t) = -a(t)x(t) + b(t)g(x(t - r(t))) \tag{3.1}$$

presents severe challenges when we attempt to show that solutions tend to 0 using Liapunov functionals. We are interested in the case where a can be negative some of the time, a and b are related on average, both a and r' can be unbounded. The problem of boundedness of r' is discussed in Knyazhishche-Shcheglov [14] and Yoshizawa [20], for example. Seifert [17] points out the need for $t - r(t)$ to tend to ∞.

Here, we ask that a, b, and r be continuous, that

$$\int_0^t a(s)ds \to \infty \text{ as } t \to \infty, \tag{3.2}$$

$$\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds \le \alpha < 1, t \ge 0, \tag{3.3.}$$

$$0 \le r(t), t - r(t) \to \infty \text{ as } t \to \infty, \tag{3.4}$$

there is an $L > 0$ so that if $|x|$, $|y| \le L$ then

$$g(0) = 0 \text{ and } |g(x) - g(y)| \le |x - y|. \tag{3.5.}$$

THEOREM 3.1. If (3.2)-(3.5) hold, then every solution of (3.1) with small continuous initial function tends to 0 as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.

PROOF. For the α and L, find $\delta > 0$ with $\delta + \alpha L \le L$. Let $\psi : (-\infty, 0] \to R$ be a given continuous function with $|\psi(t)| < \delta$ and let

$$S = \{\phi : R \to R | \|\phi\| \le L, \phi(t) = \psi(t) \text{ if } t \le 0, \phi(t) \to 0 \text{ as } t \to \infty, \phi \in C\}$$

where $\|\cdot\|$ is the supremum norm.

Define $P : S \to S$ by

$$(P\phi)(t) = \psi(t) \text{ if } t \le 0$$

and

$$(P\phi)(t) = e^{-\int_0^t a(s)ds}\psi(0) + \int_0^t e^{-\int_s^t a(u)du} b(s)g(\phi(s - r(s)))ds, t \ge 0.$$

Clearly, $P\phi \in C$. We now show that $(P\phi)(t) \to 0$ as $t \to \infty$. Let $\phi \in S$ and $\epsilon > 0$ be given. Then $\|\phi\| \le L$, there exists $t_1 > 0$ with $|\phi(t - r(t))| < \epsilon$ if $t \ge t_1$, and there exists $t_2 > t_1$ such that $t > t_2$ implies that $e^{-\int_{t_1}^t a(u)du} < \epsilon/(L\alpha)$.

Then $t > t_2$ implies that

$$\begin{aligned}
&\left|\int_0^t e^{-\int_s^t a(u)du} b(s)g(\phi(s - r(s)))ds\right| \\
&\le \int_0^{t_1} e^{-\int_s^t a(u)du}|b(s)|Lds + \int_{t_1}^t e^{-\int_s^t a(u)du}|b(s)|\epsilon ds \\
&\le e^{-\int_{t_1}^t a(u)du} \int_0^{t_1} e^{-\int_s^{t_1} a(u)du}|b(s)|Lds + \alpha\epsilon \\
&\le \alpha L e^{-\int_{t_1}^t a(u)du} + \alpha\epsilon \\
&\le \epsilon + \alpha\epsilon.
\end{aligned}$$

To see that P is a contraction under the supremum norm, if $\phi, \eta \in S$, then

$$|(P\phi)(t) - (P\eta)(t)| \le \int_0^t e^{-\int_s^t a(u)du}|b(s)| \|\phi - \eta\| ds \le \alpha \|\phi - \eta\|$$

with $\alpha < 1$ by (3.3).

Hence, for each such initial function, P has a unique fixed point in S which solves (3.1) and tends to 0.

To get stability for solutions starting at $t_0 = 0$, let $\epsilon > 0$ be given and do the above work for $L = \epsilon$.

4. A Fully Nonlinear Stability Result

Consider the scalar equation

$$x'(t) = -a(t)x^3 + g(t, x^2, x) \tag{4.1}$$

where

$$a(t) \geq 0, \int_0^\infty a(t)dt = \infty, g(t, y, 0) = 0,$$

and

$$|g(t, y, x) - g(t, y, w)| \leq b(t)|y||x - w|. \tag{4.2}$$

Now we come to a recurring problem. We require that for each bounded continuous function $z^2(t)$ with $z^2(t) \geq c$ for some $c > 0$, there is an $\alpha < 1$ with

$$\int_0^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)ds \leq \alpha, t \geq 0.$$

An obvious sufficient condition is that $a(t) \geq kb(t)$ and $a(t) \geq 0$ for all t and some $k > 1$. But that is much too severe. We would like for a to be zero on long intervals when b is nonzero.

But here we come to a real difficulty. In these fully nonlinear problems we will use the unknown exact solution as part of the mapping. Thus, we need to rely on additional information to ensure that solutions exist on $[t_0, \infty)$. An example of (4.1) is

$$x' = -a(t)x^3 + b(t)x^3.$$

If $a(t) < b(t)$ and $b(t) > 0$ on any interval $[t_0, t_1]$, however short, there are solutions with finite escape time. Hence, it will be necessary to work with particular initial times t_0 for which

$$\int_{t_0}^t [-a(s) + b(s)]ds \leq 0 \text{ for all } t \geq t_0. \tag{$*$}$$

Obviously, if $a(t) \geq b(t)$ for all t, this would hold for any t_0.

LEMMA. Let (*) hold and let $x_0 \in R$. Then $x(t, t_0, x_0)$ is defined for all $t \geq t_0$.

PROOF. Let $x(t) = x(t, t_0, x_0)$ be a solution of (4.1) with maximal interval of definition $[t_0, t_1)$. It is known that $t_1 = \infty$ or $\lim_{t \to t_1} |x(t)| = \infty$. Then define a Liapunov function

$$V(x) = |x|$$

so that along the solution we have

$$V'(x(t)) \leq -a(t)|x(t)|^3 + b(t)|x(t)|^3 = [-a(t) + b(t)]V^3.$$

If we separate variables and integrate we obtain

$$-V(t)^{-2} + V(t_0)^{-2} \leq 2\int_{t_0}^{t} [-a(s) + b(s)]ds \leq 0$$

so that $|x(t)|^3 \leq |x(t_0)|^3$, a contradiction to the finite escape time.

We now assume that for a given x_0 and a given $c < |x_0|$, there is an $\alpha < 1$ such that if $z^2(t)$ is continuous and $x_0^2 \geq z^2(t) \geq c$, then

$$\int_{t_0}^{t} e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)ds \leq \alpha \text{ for } t \geq t_0. \tag{4.3}$$

LEMMA. Suppose that if $t_n \to \infty$ and if

$$K_n = \sup_{t \geq t_n} \int_{t_n}^{t} [-a(s) + b(s)]ds, \text{ then } \lim_{n\to\infty} K_n = 0. \tag{**}$$

If $x(t)$ is a solution of (4.1) on $[t_0, \infty)$ either $x(t) \to 0$ or there is a $c > 0$ with $|x(t)| \geq c$.

PROOF. Suppose there is a sequence $t_n \to \infty$, a sequence $s_n > t_n$, and a $c > 0$ with $|x(t_n)| \to 0$ and $|x(s_n)| = c$. Rename indices so that $|x(t_n)| < c/2$ for all n. Using the V as in the proof of the first lemma, and taking

$$k_n = \int_{t_n}^{s_n} [-a(s) + b(s)]ds$$

we have, upon integration of V', the relation

$$-V^{-2}(s_n)) + V^{-2}(t_n)) \leq 2k_n.$$

If some $k_n \leq 0$, then $V(s_n) \leq V(t_n)$, a contradiction. Hence, $k_n \geq 0$ for all n and $k_n \leq K_n \to 0$ as $n \to \infty$ so

$$V^{-2}(t_n)) - 2k_n \leq c^{-2},$$

a contradiction since $V(t_n) \to 0$.

In particular, from this result, for each x_0 there is a solution $x(t, 0, x_0) =: z(t)$ defined on $[0, \infty)$. If we strengthen the Lipschitz condition in (4.2) it is unique. We will now outline the method to be used on these problems.

1. We shall assume or prove that there is a t_0 and for each x_0 there is a unique solution $x(t, t_0, x_0) =: z(t)$ on $[t_0, \infty)$. When (*) and (**) hold we have shown in the lemmas how this might be done in a particular problem.

2. Hence, $z(t)$ is the unique solution of

$$x' = -a(t)x^2(t)x + g(t, z^2(t), x), x(t_0) = x_0. \tag{4.4}$$

3. PROBLEM. In what space does $z(t)$ lie? We want to show that it lies in

$$S = \{\phi : [t_0, \infty) \to R | \phi(t_0) = x_0, \phi(t) \to 0 \text{ as } t \to \infty, \phi \in C\}. \tag{4.5}$$

Here, $\|\cdot\|$ will denote the supremum norm.

The unique solution of (4.4) is

$$x(t) = x_0 e^{-\int_{t_0}^t a(s)z^2(s)ds} + \int_{t_0}^t e^{-\int_s^t a(u)z^2(u)du} g(s, z^2(s), x(s))ds. \tag{4.6}$$

4. Define $P : S \to S$ by

$$(P\phi)(t) = x_0 e^{-\int_{t_0}^t a(s)z^2(s)ds} + \int_{t_0}^t e^{-\int_s^t a(u)z^2(u)du} g(s, z^2(s), \phi(s))ds. \tag{4.7}$$

5. If P has a fixed point, it is z, and so $z \in S$ which means that $z(t) \to 0$.
6. In this example, when (*) and (**) hold we know from the lemmas that either:
a) $z(t) \to 0$, so there is nothing to prove, or
b)$|z(t)| \geq c > 0$ so $\int_0^\infty a(t)z^2(t)dt = \infty$.
Thus, we assume that b) holds.
7. Clearly, $(P\phi)(t_0) = x_0$ and $P\phi \in C$. We now show that $(P\phi)(t) \to 0$ as $t \to \infty$ and we take $t_0 = 0$ for brevity.

Let $\epsilon > 0$ and $\phi \in S$ be given and let $c > 0$ be found. Find t_1 so that $|\phi(t)| < \epsilon/2$ if $t \geq t_1$. Then using (4.3) we obtain

$$\begin{aligned} &\int_0^t e^{-\int_s^t a(u)z^2(u)du} |g(s, z^2(s), \phi(s))|ds \\ &\leq \int_0^{t_1} e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)|\phi(s)|ds + \int_{t_1}^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)|\phi(s)|ds \\ &\leq e^{-\int_{t_1}^t a(u)z^2(u)du} \int_0^{t_1} e^{-\int_s^{t_1} a(u)z^2(u)du} b(s)z^2(s)\|\phi\|ds \\ &\quad +(\epsilon/2) \int_0^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)ds \\ &\leq \alpha\|\phi\|e^{-\int_{t_1}^t a(u)z^2(u)du} + (\epsilon/2)\alpha. \end{aligned}$$

The first term tends to zero as $t \to \infty$ and the second term can be made as small as we please.

8. To see that we have a contraction, for $\phi, \psi \in S$ we have

$$\begin{aligned} |(P\phi)(t) - (P\psi)(t)| &\leq \int_{t_0}^t e^{-\int_s^t a(u)z^2(u)du} |g(s, z^2(s), \phi(s)) - g(s, z^2(s), \psi(s))|ds \\ &\leq \int_{t_0}^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)|\phi(s) - \psi(s)|ds \\ &\leq \alpha\|\phi - \psi\|. \end{aligned}$$

Thus, P does have a fixed point and it is in S.

Further work on stability by fixed point theory may be found in [9] and [10] using Schauder's theorem, Krasnoselskii's theorem, and an extension of Krasnoselskii's theorem given in [2].

5. A Direct Stability Mapping

We arrive now at the fundamental problem. We have seen how simple it is to produce direct fixed point mappings for existence theory. We have devised a simple method of direct fixed point maps for periodic solutions. And we have presented fixed point maps for stability problems under two types of variation of parameter transformations. Our basic problem is to devise an effective way to define direct fixed point mappings for stability problems.

Given an equation

$$x' = f(t, x)$$

with the foreknowledge that all solutions tend to zero, we can write its integral equation as

$$x(t) = -\int_t^\infty f(s, x(s))ds.$$

This will be true for every solution and no initial condition is specified.

We might begin as follows. Let

$$S = \{\phi : [0, \infty) \to R || \int_0^\infty \phi(t)dt| < \infty, \phi \in C\}.$$

For a fixed $x_0 \neq 0$ and for each $\phi \in S$ such that $\int_0^\infty \phi(t)dt \neq 0$, define

$$(P\phi)(t) = (1/k)f(t, -\int_t^\infty k\phi(s)ds)$$

where

$$-k\int_0^\infty \phi(s)ds = x_0.$$

If P has a fixed point ϕ, then

$$k\phi(t) = f(t, -k\int_t^\infty \phi(s)ds)$$

and

$$-k\int_t^\infty \phi(s)ds$$

solves the initial value problem.

Clearly, conditions must be added to ensure that P maps S into itself. S must be further specified, depending on the type of stability sought. And we must select the proper fixed point theorem.

But successful completion of the problem will be a significant contribution to the theory.

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