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FIXED POINT THEOREMS IN ORDERED SETS AND APPLICATIONS

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Abstract. The first aim of this paper is to generalize some fixed point results concerning fixed points in ordered sets where we continue the investigations of [6]. Then, we will present some results concerning differential equations where we use the Tarski theorem.
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1. Introduction

There are three fundamental fixed point principles which hold on ordered structures: The Zermelo Theorem [30] for operators which satisfies the property $x \leq f(x)$, the Tarski Theorem for isotone maps [19] [9] and the Kantorovich Theorem for continuous maps [9].
All the above principles are independent of the Axiom of Choise.
We remark that many results in metric fixed point theory can be also provided the variant of Tarski Theorem showed by Amann [5].
In this paper we present some results concerning ordered sets with monotone operators and later we state other monotonicity conditions and we apply the Tarski theorem to differential equations.

2. The Tarski principle

Let P a partially ordered set (poset) (i.e. a set with reflexive, antisymmetric and transitive relation $\leq$), 0 and 1 being its least and greatest elements (if they exists). Let X a subset of a poset P. An element $y \in P$ is an *upper (lower) bound* of X iff $x \leq y$ ($y \leq x$) for all $x \in X$.
The terms the least upper bound and the greatest lower bound will be abbreviated to *sup* and *inf*, respectively.
A non empty subset X of an ordered set P is called a *chain* if is a totally ordered, that is, for every pair $x, y \in X$ we have $x \leq y$ *or* $y \leq x$. Let P be a poset.
For $f : P \to P$ and $x \in P$,

$$O_f(x) := \{x,\ f(x),\ f^2(x), ...,\ f^n(x), ...\}$$

is called the *orbit of x.*
For any mapping $f : P \to P$ an element $x \in P$ is called fixed point of f if $x = f(x)$ and we write F_f for the fixed point set.
The two fundamental results the following theorems:

Theorem 2.1. *(Tarski) Let $(L, \leq)$ be a complete lattice and $f : L \longrightarrow L$ increasing. Then f has a fixed point.*

Theorem 2.2. *(Knaster-Tarski) Let $(P, \leq)$ be a partially ordered set and $f : P \longrightarrow P$ increasing. Assume that there is a $x \in P$ such that $x_0 \leq f(x_0)$ and every chain in $\{x \in P \mid x_0 \leq x\}$ has a supremum. Then f has a fixed point.*

3. Main Results

Theorem 3.1. *Let $(P, \leq)$ be a poset such that every nonempty chain has upper bound, $f : P \longrightarrow P$, $I \subset N^*$ a finite set of natural numbers and $K = \{k_i \mid i \in I\} \subset N^*$ contains two relatively prime numbers. If $\{f^{k_i}(x) \mid i \in I\}$ totally ordered and*

$$x \leq \min \left\{ f^{k_i}(x) \mid i \in I \right\}$$

for all $x \in P$ then $F_f \neq \emptyset$.

Proof. From Zorn lema we have that $Max(P) \neq \emptyset$, so this set have least element m. For m we have:

$$m \leq \min \left\{ f^{k_i}(m) \mid i \in I \right\}$$

But the set $\left\{ f^{k_i}(m) \mid i \in I \right\}$ totally ordered, so

$$m \leq f^{k_i}(m) \quad \forall i \in I$$

and the maximality of m implied

$$f^{k_i}(m) = m, \quad \forall i \in I.$$

Because exists $i, j \in I$ such that $(k_i, k_j) = 1$, we have natural numbers n and l which satisfied the condition

$$nk_i - lk_j = 1.$$

From this

$$m = f^{nk_i}(m) = f^{1+lk_j} = f\left(f^{lk_j}(m)\right) = f(m)$$

which means that m fixed point for f.

□

Remarks.
1. The proof implies

$$Max(P) \subset F_f.$$

2. If K not contains relatively prime numbers, we have

$$f^{d_{ij}}(m) = m$$

wherw $d_{ij} = (k_i, k_j)$ and $m \in Max(P)$.
3. This theorem implies Theorem 2.1 and Theorem 2.2.

Corollary 3.1. *(Amann [5])*
Let $(P, \leq)$ chain complet poset and $f : P \longrightarrow P$ be an increasing operator. If exist $a, b \in P$ such that

$$a \leq b,\ a \leq f(a),\ f(b) \leq b.$$

Then f has a least and a greatest fixed point in $[a, b]$.

Corollary 3.2. *(Taskovic [25]) Let $(P, \leq)$ be a poset such that exists $a,\ b \in P$ for which*

$$a \leq f(a) \leq f(b) \leq b.$$

and $f : P \longrightarrow P$ be an increasing operator. If P conditionally complete, then $F_f \neq \emptyset$.

Theorem 3.2. *Let $(P, \leq)$ be a poset such that every chain has an upper bound, $f : P \longrightarrow P$ an increasing operator, $I \subset N^*$ a finite set of natural numbers and $K = \{k_i \mid i \in I\} \subset N^*$ contains two relatively prime numbers. If $\{f^{k_i}(x) \mid i \in I\}$ totally ordered and exists x_0 such that*

$$x_0 \leq \min\left\{f^{k_i}(x_0) \mid i \in I\right\},$$

then $F_f \neq \emptyset$.

Proof. Let

$$A = \left\{x \mid x \leq \min\left\{f^{k_i}(x) \mid i \in I\right\}\right\}.$$

We observe that A is not empty ($x_0 \in A$), and because f increasing $f(A) \subset A$.
Let one chain C of A. Then C has a supremum c. We prove that $c \in A$. Let $x \in C$,

$$x \leq \min\left\{f^{k_i}(x) \mid i \in I\right\}$$

$$x \leq c$$

Because f increasing operator

$$f^{k_i}(x) \leq f^{k_i}(c) \quad \forall i \in I$$

$$x \leq \min\left\{f^{k_i}(x) \mid i \in I\right\} \leq \min\left\{f^{k_i}(c) \mid i \in I\right\}, \quad \forall x \in C$$

We take the supremum

$$c \leq \min\left\{f^{k_i}(c) \mid i \in I\right\}.$$

Which mean that c an element of A.
From Theorem 3.1 we have that f has a fixed point.

□

Now we try that change in the condition of Theorem 3.1 and Theorem 3.2 the minimum to maximum. We have the following result.

Theorem 3.3. *Let $(P, \leq)$ be a poset such that every nonempty chain has upper bound, $f : P \longrightarrow P$, $I \subset N^*$ a finite set of natural numbers and $K = \{k_i \mid i \in I\} \subset N^*$. If $\{f^{k_i}(x) \mid i \in I\}$ totally ordered and*

$$x \leq \max\left\{f^{k_i}(x) \mid i \in I\right\}$$

for all $x \in P$ then exists $i \in I$ such that $F_{f^{k_i}} \neq \emptyset$.

Proof. By the Zorn maximality principle we have that $Max(P) \neq \emptyset$, so they have one element m. For m

$$m \leq \max\left\{f^{k_i}(m) \mid i \in I\right\}$$

But $\left\{f^{k_i}(m) \mid i \in I\right\}$ is totally ordered which implied that exists $i \in I$, such that

$$m \leq f^{k_i}(m)$$

By the maximality of m

$$f^{k_i}(m) = m \quad .$$

□

Theorem 3.4. *Let $(P, \leq)$ be a poset such that every nonempty chain has upper bound, $f : P \longrightarrow P$, $I \subset N^*$ a finite set of natural numbers and $K = \{k_i \mid i \in I\} \subset N^*$. If $\{f^{k_i}(x) \mid i \in I\}$ totally ordered and exists x_0 such that*

$$x_0 \leq \max\left\{f(x_0),\ f^{k_i}(x_0) \mid i \in I\right\}$$

then $F_f \neq \emptyset$.

Proof. Let

$$A_1 = \{x \mid x \leq f(x)\}\,.$$

If A_1 nonempty from Corollary 3.1 we have $F_f \neq \emptyset$.
In the next we suppose that $A_1 = \emptyset$.
Let

$$f^n(x) = \max\left\{f^{k_i}(x_0) \mid i \in I\right\}$$

and

$$A_2 = \{x \mid x \leq f^n(x)\}\,.$$

Because A_1 nonempty and

$$x_0 \leq \max\left\{f(x_0),\ f^{k_i}(x_0) \mid i \in I\right\}$$

we have

$$f(x_0) \leq x_0 \leq f^n(x_0)$$

so A_2 nonempty ($x_0 \in A_2$).
The monotonicity of f implied that A_2 is f^n invariant. We apply Corollary 3.1, where we consider f^n instead f. So for maximal element m we have $m \in F_{f^n}$ or

$$f^n(m) = m.$$

But $f(m) \leq m$ and f^{n-1} increasing, which implies

$$m = f^n(m) \leq f^{n-1}(m)$$

From the maximality of m follows that $m = f^{n-1}(m)$. We proved that

$$m \in F_{f^n} \cap F_{f^{n-1}}.$$

which implied that $m \in F_f$.

□

4. Application

In this section we study the existence and uniqueness of the bounded solution of a boundary value problem. The proof is based on the Tarski theorem concerning the existence of the fixed points in complete lattice.
We consider the following problem

$$\begin{cases} \frac{d^2u}{dt^2} = \varphi(u,t) \\ \\ u(0) = -\alpha,\ \alpha > 0 \end{cases} \tag{4.1}$$

where α is a given positive number.
We remark that if

$$\varphi(u,t) = t(e^{\frac{u}{t}} - 1)$$

the problem is representing the motion of a particle in an ionized field under the influence of the Ukawa potential.
Gross in [10] obtained some conditions for the uniqueness of the solution.
In the study of the problem we apply the theorem of Tarski concerning the existence of the fixed point in complete lattice.
Gross obtained the following result.

Theorem 4.1. *(Gross* [10]*) If φ satisfies the following conditions:*
1)

$$\varphi \in C(\mathbb{R}_- \times \mathbb{R}_+) \ \ \textit{and} \ \ \frac{\partial f}{\partial u} \in C(\mathbb{R}_- \times \mathbb{R}_+)$$

2)

$$\varphi(0,s) = 0 \ \ \textit{and} \ \ \varphi(u,s) \le 0 \quad \forall u \in \mathbb{R}_-,\ s \in \mathbb{R}_+$$

and is nondecreasing as a function of u for every $t > 0$.
3)

$$\varphi(u,t) - u \ge 0$$

and is nonincreasing as a function of u,
then there exist a unique, negativ, bounded solution of (4.1) in $C^2(\mathbb{R}_+^*)$. This solution is nondecreasing and concave and tends to some positive constant, as $r \longrightarrow \infty$.

We first establish the existence.
Let

$$X = \{u \mid u : R_+^* \longrightarrow R,\ -\alpha \le u(t) \le -\alpha\tfrac{f(t)}{f(0)} \text{ and } \tfrac{u(t)}{f(t)} \text{ is nonincreasing in } t\}$$

We introduce the following ordering relation:

$$u_1 \le u_2 \ \text{ iff } u_1(t) \le u_2(t), \quad \forall t \in R_+^*.$$

X is a complete lattice.
If we apply the Tarski theorem (Theorem 2.1) to

$$(T_1u)(t) = -\alpha\frac{f(t)}{f(0)} - f(t)\int_0^t \frac{1}{f^2(r)} \int_r^\infty [f(s)\varphi(u(s),s) - u(s)f''(s)]\, ds\, dr$$

we have the following result:

Theorem 4.2. *(Bege [7]) If φ in the problem 4.1 satisfies:*
a)

$$\varphi \in C(\mathbb{R}_- \times \mathbb{R}_+^*) \ \ and \ \ \frac{\partial \varphi}{\partial u} \in C(\mathbb{R}_- \times \mathbb{R}_+^*)$$

b)

$$\varphi(0,s) = 0 \ \ and \ \ \varphi(u,s) \leq 0 \quad \forall u \in \mathbb{R}_-, \ s \in \mathbb{R}_+$$

c)
There exist the function $f : \mathbb{R}_+ \longrightarrow (0,1]$ such that

$$f \in C^2(\mathbb{R}_+), \ f''(s) \geq 0, \ \lim_{s\to\infty} f'(s) = 0, \ f(s) \leq f(0) \quad \forall s \in \mathbb{R}_+$$

for which

$$\varphi'_u(u,s) \leq \frac{f''(s)}{f(s)} \quad u \in \mathbb{R}_-, \ s \in \mathbb{R}_+.$$

The problem has a bounded solution in $C^2(\mathbb{R}_+)$.

If we consider

$$(T_2 u)(r) = -\alpha - \int_0^t dr \int_r^\infty \varphi(s, u(s))\, ds.$$

and we apply a uniqueness theorem (see [17]) for a boundary value problem on the finite interval to obtain the following uniqueness result

Theorem 4.3. *(Bege [7]) If φ in the problem 4.1 satisfies:*
a)

$$\varphi \in C(\mathbb{R}_- \times \mathbb{R}_+^*) \ \ and \ \ \frac{\partial \varphi}{\partial u} \in C(\mathbb{R}_- \times \mathbb{R}_+^*).$$

b)

$$\varphi(0,s) = 0 \ \ and \ \ \varphi(u,s) \leq 0, \quad \forall u \in \mathbb{R}_-, \ s \in \mathbb{R}_+$$

c)
there exist the function $f : \mathbb{R}_+ \longrightarrow (0,1]$ such that

$$f \in C^2(\mathbb{R}_+), \ f''(s) \geq 0, \ \lim_{s\to\infty} f'(s) = 0, \ f(s) \leq f(0), \quad \forall s \in \mathbb{R}_+$$

for which

$$\varphi'_u(u,s) \leq \frac{f''(s)}{f(s)}, \quad u \in \mathbb{R}_-, \ s \in \mathbb{R}_+.$$

d)
$\varphi(u,t)$ is nondecreasing function of u for all $t > 0$
then there exists a unique , negative, bounded solution $u \in C^2(\mathbb{R}_+)$ to 4.1.

Remark.
If

$$f(t) = e^{-t}$$

then f satisfies c) and we obtain the Gross theorem 4.1 ([10]).

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