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PAIRS OF CLASSES OF TOPOLOGICAL SPACES WITH THE FIXED POINT PROPERTY

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Abstract. There have appeared many generalizations of the Kakutani-Fan-Glicksberg fixed point theorem. Motivated by these generalizations we introduce the concept of fixed point property for a pair $(\mathcal{T}, \mathcal{C})$ of classes of compact Hausdorff topological spaces; section properties and minimax inequalities are given.
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1. Introduction

A *map* (or a *multifunction*) $T : X \to Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the values $T(x) \subset Y$. Given two maps $S : X \to Y$, $T : Y \to Z$ the *composite* $T \circ S : X \to Z$ is defined by $(T \circ S)x = T(Sx) = \cup\{Ty : y \in Sx\}$.

Let X and Y be topological spaces. A map $T : X \to Y$ is said to be *upper semicontinuous* if for each closed set $F \subset Y$ the *lower inverse* of F under T, that is $T^{-1}(F) = \{x \in X : T(x) \cap F \neq \emptyset\}$, is a closed subset of X or, equivalently, if for each open set $G \subset Y$ the *upper inverse* of G under T, that is $T^{+1}(G) = \{x \in X : T(x) \subset G\}$, is an open subset of X. Note that if Y is compact Hausdorff and Tx is closed for each $x \in X$, then T is upper semicontinuous if and only if the *graph* of T, that is $\{(x, y) \in X \times Y : y \in T(x)\}$, is closed in $X \times Y$. Recall also that the composite and the product of two upper semicontinuous maps are upper semicontinuous too.

For a class of sets $\mathcal{C}$ and a set X we shall denote by

$$\mathcal{C}(X) = \{C \in \mathcal{C} : C \subset X\} \text{ and } \mathcal{C}^*(X) = \{C \in \mathcal{C} : \emptyset \neq C \subset X\}.$$

We say that a map $T : X \to Y$ has $\mathcal{C}$ (resp. $\mathcal{C}^*$) values if for each $x \in X$, $T(x) \in \mathcal{C}(X)$ (resp. $T(x) \in \mathcal{C}^*(X)$).

We say that an ordered pair $(\mathcal{T}, \mathcal{C})$ consisting of two classes of compact Hausdorff topological spaces has the *fixed point property* provided:

(i) $X, Y \in \mathcal{T} \Rightarrow X \times Y \in \mathcal{T}$;
(ii) $C \in \mathcal{C}(X), D \in \mathcal{C}(Y) \Rightarrow C \times D \in \mathcal{C}(X \times Y)$, for each $X, Y \in \mathcal{T}$;

(iii) for each $X \in \mathcal{T}$ any upper semicontinuous map $T : X \to X$ with $\mathcal{C}^*$ values has a fixed point

Four examples of pairs $(\mathcal{T}, \mathcal{C})$ having the fixed point property will be given in the sequel.

Ex.1. $\mathcal{T}$ and $\mathcal{C}$ are both the class of all compact convex subsets of all Hausdorff locally convex topological vector spaces. In this case condition (iii) is satisfied according to the Kakutani-Fan-Glicksberg fixed point theorem (see [1], [2], [3])

Ex.2. $\mathcal{T}$ is the class of all compact convex subsets of all Hausdorff locally convex topological vector spaces and for each $X \in \mathcal{T}$, $\mathcal{C}(X)$ consists of all compact acyclic subsets of X (recall that a topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish). The product of two acyclic sets is of course acyclic by the Kunneth formula (see [5]) and condition (iii) is satisfied according to Theorem 7 in [6].

In order to give two other examples of pairs $(\mathcal{T}, \mathcal{C})$ with the fixed point property we shall recollect some definitions introduced by S. Park and H. Kim (see [8] and [4]). For a set X we shall denote by $\langle X\rangle$ the set of all nonempty finite subsets of X.

A *generalized convex space* or a *G-convex space* (X, Γ) consists of a topological space X and a function $\Gamma : \langle X\rangle \to X$ such that:

(a) $A, B \in \langle X\rangle$, $A \subset B \Rightarrow \Gamma_A = \Gamma(A) \subset \Gamma_B$; and
(b) for each $A \in \langle X\rangle$ with $|A| = n + 1$ there exists a continuous function $\Phi_A : \Delta_n \to \Gamma_A$ such that $J \in \langle A\rangle$ implies $\Phi_A(\Delta_J) \subset \Gamma_J$ (here Δ_n denotes the standard n-simplex and Δ_J denotes the face of Δ_n coresponding to $J \in \langle A\rangle$).

For an $(X; \Gamma)$ a subset C of X is said to be G-convex if $A \in \langle C\rangle$ implies $\Gamma_A \subset C$.

A G-convex space $(X; \Gamma)$ is called:

(a) *locally G-convex uniform space* if it satisfies the following conditions:
  (a_1) X is a Hausdorff uniform space with the basis $\mathcal{V}$;
  (a_2) for each $V \in \mathcal{V}$ and $x \in X$ the set $\{x' \in X : (x, x') \in V\}$ is G-convex.
(b) *of type II* if it is separated and satisfies the following conditions:
  (b_1) for each $x \in X$, $\{x\}$ is G-convex;
  (b_2) for any compact G-convex subset Y of X and each open neighborhood V of Y there exists an open neighborhood U of Y such that $\cap\{Z : U \subset Z \subset X$ and Z is G-convex $\} \subset V$

Ex.3. $\mathcal{T}$ is the class of all compact locally G-convex uniform spaces and for each $X \in \mathcal{T}$, $\mathcal{C}(X)$ consists of all compact G-convex subsets of X. In this case condition (iii) is satisfied according to Lemma 4 and Theorem 4 in [7].

Ex.4. $\mathcal{T}$ is the class of all compact G-convex spaces of type II and for each $X \in \mathcal{T}$, $\mathcal{C}(X)$ consists of all compact G-convex subsets of X. In this case condition (iii) is satisfied according to Theorem 2 in [4].

2. Section properties

In the sequel let us fix a pair $(\mathcal{T}, \mathcal{C})$ having the fixed point property.

Theorem 2.1. *Let $X, Y \in \mathcal{T}$. Then for every two maps $S : X \to Y$, $T : Y \to X$ with $\mathcal{C}^*$ values the composite $T \circ S$ has a fixed point.*

Proof. Consider the diagram

$$X \times Y \xrightarrow{p} Y \times X \xrightarrow{T\times S} X \times Y$$

where $p(x, y) = (y, x)$ and $(T \times S)(y, x) = T(y) \times S(x)$. The map $(T \times S) \circ p$ is upper semicontinuous and by (ii) it takes $\mathcal{C}^*$ values. By (iii) $(T \times S) \circ p$ has a fixed point, i.e. for some $(x_0, y_0) \in X \times Y$ we have $(x_0, y_0) \in (T \times S)(y_0, x_0)$. Hence $x_0 \in T(y_0)$, $y_0 \in S(x_0)$ and consequently $x_0 \in (T \circ S)(x_0)$. □

As a direct consequence of Theorem 2.1 we have

Theorem 2.2. *Let $X, Y \in \mathcal{T}$ and M, N be two open subsets of $X \times Y$ such that $M \cup N = X \times Y$. Suppose that the following conditions are satisfied:*

(2.1) *For each $x \in X$, $\{y \in Y : (x, y) \notin M\} \in \mathcal{C}(Y)$.*
(2.2) *For each $y \in Y$, $\{x \in X : (x, y) \notin N\} \in \mathcal{C}(X)$.*

Then at least one of the following assertions holds:

(a) *There exists a point $x_0 \in X$ such that $\{x_0\} \times Y \subset M$.*
(b) *There exists a point $y_0 \in Y$ such that $X \times \{y_0\} \subset N$.*

Proof. Let $M' = (X \times Y)\backslash M$ and $N' = (X \times Y)\backslash N$. Define $S : X \to Y$, $T : Y \to X$ by putting

$$S(x) = \{y \in Y : (x, y) \in M'\}, \;\; T(y) = \{x \in X : (x, y) \in N'\}.$$

Since M' is closed in $X \times Y$, each $S(x)$ is closed in Y and the graph of S is closed in $X \times Y$. Hence S is upper semicontinuous and by (2.1) it follows that S takes $\mathcal{C}$ values. Similarly we can prove that T is upper semicontinuous and takes $\mathcal{C}$ values.

Suppose that both assertions (a) and (b) are not true. Then for each $x \in X$ there exists $y \in Y$ such that $(x, y) \in M'$, that is S has $\mathcal{C}^*$ values and similarly T has $\mathcal{C}^*$ values. By Theorem 2.1, $T \circ S$ has a fixed point, or equivalently there exists $(x_0, y_0) \in X \times Y$ such that $y_0 \in S(x_0)$ and $x_0 \in T(x_0)$. Then $(x_0, y_0) \in M' \cap N'$ which contradicts $M \cup N = X \times Y$. □

Corollary 2.3. *Let $X, Y \in \mathcal{T}$ and N be an open subset of $X \times Y$ satisfying:*

(2.3) *There exists a upper semicontinuous map $T : X \to Y$ with $\mathcal{C}^*$ values such that graph$T \subset N$.*
(2.4) *For each $y \in Y$, $\{x \in X : (x, y) \notin N\} \in \mathcal{C}(X)$.*

Then there exists a point $y_0 \in Y$ such that $X \times \{y_0\} \subset N$.

Proof. Consider the set $M = X \times Y \setminus \text{graph}T$. From hypothesis it readily follows that:

$$\begin{cases} M \text{ is an open subset of } X \times Y; \\ \text{for each } x \in X,\ \{y \in Y : (x, y) \notin M\} \in \mathcal{C}(Y); \\ \text{for each } x \in X,\ \{x\} \times Y \not\subset M. \end{cases}$$

Moreover $M \cup N = X \times Y$. The conclusion follows from Theorem 2.2 □

Corollary 2.4. *Let $X \in \mathcal{T}$ and M be an open subset of $X \times X$ satisfying:*

(2.5) $\Delta = \{(x, x) : x \in X\} \subset M$.
(2.6) *For each $x \in X$, $\{y \in X : (x, y) \notin M\} \in \mathcal{C}(X)$.*

Then there exists a point $x_0 \in X$ such that $\{x_0\} \times X \subset M$.

Proof. Apply Theorem 2.2 in the case $Y = X$, $N = X \times X \setminus \Delta$ and observe that the assertion (b) in the conclusion of this theorem cannot take place. □

3. Minimax inequalities

Let $X \in \mathcal{T}$. A function $f : X \to \mathbb{R}$ will be called *$\mathcal{C}$-quasiconcave* if for each $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x) \geq \lambda\} \in \mathcal{C}(X)$ and *$\mathcal{C}$-quasiconvex* if $-f$ is $\mathcal{C}$-quasiconcave.

Theorem 3.1. *Let $X, Y \in \mathcal{T}$ and $f, g : X \times Y \to \mathbb{R}$ two real valued functions satisfying:*

(3.1) $f \leq g$.
(3.2) *f is upper semicontinuous and g is lower semicontinuous on $X \times Y$.*
(3.3) *For each $x \in X$, $f(x, \cdot)$ is $\mathcal{C}$-quasiconcave on Y.*
(3.4) *For each $y \in Y$, $g(\cdot, y)$ is $\mathcal{C}$-quasiconvex on X.*

Then, given any $\alpha, \beta \in \mathbb{R}$ $\beta < \alpha$, at least one of the following assertions holds:

(a) *There exists $x_0 \in X$ such that $f(x_0, y) < \alpha$ for each $y \in Y$.*
(b) *There exists $y_0 \in Y$ such that $g(x, y_0) > \beta$ for each$x \in X$.*

Proof. Apply Theorem 2.2 to the sets:

$$M = \{(x, y) \in X \times Y : f(x, y) < \alpha\}, \quad N = \{(x, y) \in X \times Y : g(x, y) > \beta\}.$$

¿From the hypothesis (3.1)-(3.4) it follows readily that M, N are open in $X \times Y$, $M \cup N = X$ and assumptions (2.1)-(2.2) of Theorem 2.2 are verified. The desired result follows now from Theorem 2.2. □

Corollary 3.2. *Under the hypothesis of Theorem 3.1 the following inequality holds*

$$\inf_{x \in X} \max_{y \in Y} f(x, y) \leq \sup_{y \in Y} \min_{x \in X} g(x, y).$$

Proof. First let us observe that if f is upper semicontinuous on $X \times Y$, then for each $x \in X$, $f(x, \cdot)$ is also an upper semicontinuous function of y on Y and therefore its maximum $\max_{y \in Y} f(x, y)$ on the compact set Y exists. Similary $\inf_{x \in X} g(x, y)$ can be replaced by $\min_{x \in X} g(x, y)$.

Suppose the conclusion were false and choose two real numbers α, β such that $\sup_{y \in Y} \min_{x \in X} g(x, y) < \beta < \alpha < \inf_{x \in X} \max_{y \in Y} f(x, y)$.

We prove that neither the assertion (a) nor the assertion (b) of the conclusion of Theorem 3.1 cannot take place.

If (a) happens, then

$$\inf_{x \in X} \max_{y \in Y} f(x, y) \leq \max_{y \in Y} f(x_0, y) \leq \alpha; \text{ a contradiction.}$$

If (b) happens, then

$$\sup_{y \in Y} \min_{x \in X} g(x, y) \geq \min_{x \in X} g(x, y_0) \geq \beta; \text{ a contradiction again.}$$

□

Corollary 3.3. *Let $X \in \mathcal{T}$ and $f, g : X \times X \to \mathbb{R}$ two real-valued functions satisfying conditions (3.1)-(3.4) of Theorem 3.1. Then we have*

$$\inf_{x \in X} f(x,x) \leq \sup_{y \in X} \min_{x \in X} g(x,y).$$

Proof. We may assume that $\sup_{x \in X} f(x,x) > -\infty$. Apply Theorem 3.1 in the case $Y = X$, $\alpha = \inf_{x \in X} f(x,x)$, $\beta = \inf_{x \in X} f(x,x) - \varepsilon$, where $\varepsilon > 0$ is arbitrarily fixed. Since the assertion (a) of the conclusion of Theorem 3.1 cannot take place, it follows that there exists $y_0 \in X$ such that

$$\min_{x \in X} g(x, y_0) > \inf_{x \in X} f(x,y) - \varepsilon.$$

Clearly this implies the desired minimax inequality. □

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