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ITERATIVE APPROXIMATION OF FIXED POINTS FOR PSEUDO-CONTRACTIVE OPERATORS

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Abstract. For a nonexpansive (Liptschitzian) operator F which is not a strict contraction, the Picard iteration does not converge generally to a fixed point. By adding some pseudo-contractive type assumptions, it is possible to show that several other iterations (Krasnoselski-Schaefer, Mann, Ishikawa) converge to a fixed point of F.

The main aim of the paper is to survey some old and recent results especially related to the convergence of Krasnoselski-Schaefer iteration in the class of generalized pseudo-contractive and lipschitzian operators.

1. Introduction

Many of the most important nonlinear problems of applied mathematics reduce to finding solutions of nonlinear functional equations (integral equations, differential equations etc.) which can be formulated in terms of finding the fixed points of a given nonlinear operator of an infinite dimensional function space X into itself:

$$x = Tx \tag{1}$$

There is a classical general existence theory of fixed points for mappings satisfying compactness conditions associated with the names of Brower, Schander, Leray etc. as well as an abundant literature of metrical fixed point theorems, which establish the existence (and uniqueness) of fixed points for maps satisfying a variety of contractive conditions ([16]). The first basic result is the classical Picard-Banach-Caccioppoli principle

THEOREM 0. *Let (X, d) be a complete metric space and $T : X \to X$ a strict contraction, that is, there exists α, $0 \le \alpha < 1$ such that*

$$d(Tx, Ty) \le \alpha \cdot d(x, y) \text{ for all } x, y \in X \tag{2}$$

Then the Picard iteration (the sequence of successive approximation) (x_n), given by

$$x_n = T(x_{n-1}) = T^n(x_0),\ n = 1, 2, ... \tag{3}$$

converges to the unique fixed point x^* of T,

$$x_n \to x^* \text{ (as } n \to \infty).$$

The convergence order of the Picard iteration in Theorem 0 is given by both *a priori* and *a posteriori* estimates:

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} \cdot d(x_0, x_1),\ n \geq 1; \tag{4}$$

$$d(x_n, x^*) \leq \frac{\alpha}{1-\alpha} \cdot d(x_n, x_{n-1}),\ n \geq 1; \tag{4'}$$

This fundamental result in the fixed point theory has been extended to some larger classes of contractive and generalized contractive operators, see[16], for example, by replacing the strict contractive condition (2) by a weaker condition of the following type

$$d(Tx, Ty) \leq \varphi(d(x, y)), \qquad x, y \in X \tag{2'}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a certain compassion function or by a more general one
$d(Tx, Ty) \leq \varphi(d(x, y)), d(x, Tx), d(x, Ty), d(y, Tx), d(y, Ty)),\ x, y \in X,$

where $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ stands for a certain 5-dimensional comparison function (see [2], [21]). Important results as Kannan or Ciric theorems belong to the second class of generalized contractions [16].

If the generalized contractive condition in a such fixed point theorem is strong enough, that is, the comparison function satisfies some essential conditions, then the (possible unique) fixed point of T can be obtained by means of the sequence of successive approximations (Picard iteration).

But, if the contractive condition is slightly weaker, as for example the case whenT is only nonexpansive

$$d(Tx, Ty) \leq d(x, y),\ x, y \in X, \tag{2''}$$

then the Picard iteration (3) need no longer converge to a fixed point of T (if any).

In fact, in general, a nonexpansive operator T need not have a fixed point and even if T possesses a fixed point, the Picard iteration need not converge to this point, see Example 1 in Section 2.

Even if the fixed point can be obtained by the Picard iteration, it is of interest to determine other iteration procedures that could converge faster, in a certain sense, to the fixed point.

2. Other fixed point iteration procedures

The next example illustrates the case of a nonexpansive operator possessing an unique fixed point, for which the Picard iteration does not converge to that fixed point, except for the case when the initial approximation coincides to the fixed point itself.

EXAMPLE 1. ([9]) Let $X = [0, 1]$ and $T : [0, 1] \to [0, 1]$ be the linear function

$$T(x) = 1 - x,\ 0 \leq x \leq 1.$$

Then : a) T is nonexpansive;
b) T has an unique fixed point

$$F_T = \{x \in [0, 1]/T(x) = x\} = \{\frac{1}{2}\}.$$

c) The Picard iteration $x_n = T(x_{n-1}), n = 1, 2, ...$, yields the oscillatory sequence $a, 1 - a, a, 1 - a, a, ...$for any $x_0 = a(a \neq \frac{1}{2})$.

Taking into account the fact that the class of nonexpansive operators is very important in applications, we have to impose certain additional conditions on the ambient space or on the operator itself, in order to ensure the existence of a fixed point or to guarantee the convergence of the Picard iteration to a fixed point of the operator.

In what concern the existence problem, the following result was obtained independently by Browder, Gohde and Kirk (see [3], for example):

THEOREM 1. *Let C be a closed, bounded, and convex subset of a uniformly convex Banach space, $T : C \to C$ a nonexpansive map. Then T has a fixed point.*

The proofs of Theorem 1 are, unfortunately, all not constructive. As shown by Example 1, the Picard iteration does not converge (to the fixed point) of such a nonexpansive operator.

A similar situation is encountered for the class of lipschitzian (and pseudo-contractive, in some sense) operators, when, even if the fixed point is unique, the Picard iteration does not converge (see [1]).

To remove these difficulties we need to consider some other sequential procedures to be used for approximating fixed points.

We present here a few chronological reference points. In 1953 W. R. Mann [17] introduced an iteration procedure which can be represented in the following form

$$x_{n+1} = (1 - a_n)x_n + a_n T x_n, \quad n = 0, 1, 2, ... \tag{5}$$

where (a_n) is sequence satisfying: (i) $a_n \in [0, 1]$ and (ii) $\sum a_n = \infty$.

Mann showed that, if T is an continuous selfmap of a closed interval $[a, b]$ with at most one fixed point, then the iteration scheme (5), with $a_n = 1/(n + 1)$,converges to the fixed point of T.

In 1955, Krasnoselski [18] showed that, if X is a uniformly convex Banach space, and $T : X \to X$ is nonexpansive, then the iteration (x_n),

$$x_{n+1} = \frac{1}{2}(x_n + T x_n), \qquad n = 0, 1, 2, ... \tag{6}$$

converges to a fixed point of T.

Later, in 1957, Schaefer considered the extension of (6), by replacing 1/2 by a constant $\lambda \in [0, 1]$, that is, he introduced the iteration procedure (x_n),

$$x_{n+1} = (1 - \lambda)x_n + \lambda \cdot T x_n, \quad n = 0, 1, 2, ... \tag{6'}$$

and proved similar results [19].

It is easy to see that (x_n) given by (6') is in fact the Picard iteration for the associated operator $U = (1-\lambda)\cdot I + \lambda \cdot T$, where I is the identity.

If we take $\lambda = \frac{1}{2}$, from (6') we obtain (6), and, by putting $\lambda = 1$, from (6') we obtain the Picard iteration. Iteration (6'), which will be called in the sequel *Krasnoselski-Schaefer iteration*, is a particular case of the *Mann iteration* (5), obtained from the last one for $a_n = \lambda\,(const)$.

It was proven for continuous mappings that, if the Mann iterative process converges, then it must converge to a fixed point of T. But if T is not continuous, there is no guarantee that, even if the Mann process converges, it will converge to a fixed point of T, as shown by the next example.

EXAMPLE 2. Let $X = [0,1]$ and $T : X \to X$ the map defined by $T0 = T1 = 0$ and $Tx = 1$, for $x \in (0,1)$.

Then T is a selfmap of $[0,1]$,having the unique fixed point $x = 0$. However, the Mann iteration, with $a_n = 1/(n+1)$ and $x_0 \in (0,1)$ converges to 1, which is not a fixed point of T.

By adding some pseudocontractive hypotheses to a nonexpansive map, Ishikawa [7] proved the following result.

THEOREM 3. *Let E be a convex, compact subset of a Hilbert space H, $T : E \to E$ a lipschitzian and pseudocontractive map. Then the sequence (x_n), defined by*

$$x_{n+1} = (1-\alpha_n)\cdot x_n + \alpha_n T\ [(1-\beta_n)x_n + \beta_n T x_n],\ \ n = 0,1,2,... \tag{7}$$

where $(\alpha_n), (\beta_n)$ are sequences of positive numbers satisfying the conditions $0 \le \alpha_n \le \beta_n \le 1$, $\lim \beta_n = 0$ and $\sum \alpha_n \beta_n = \infty$, converges strongly to a fixed point of T.

The next diagram represents the most important iteration procedures considered here

$x_{n+1} = Tx_n,\ \ n \ge 0$ 1890 Picard

$\Uparrow \lambda = 1$

$x_{n+1} = \frac{1}{2}(x_n + Tx_n),\ \ n \ge 0$ 1955, Krasnoselski

$\Uparrow \lambda = \frac{1}{2}$

$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, n \ge 0, 0 \le \lambda \le 1$ 1957 (Krasnoselski-)Schaeffer

$\Uparrow a_n = \lambda(const)$

$x_{n+1} = (1-a_n)\cdot x_n + a_n \cdot Tx_n,\ n \ge 0,\ a_n \in [0,1]$ 1953 Mann

$\Uparrow b_n = 0$

$x_{n+1} = (1-a_n)\cdot x_n + a_n T\ [(1-b_n)x_n + b_n T x_n], n \ge 0, 0 \le a_n, b_n \le 1$

1974 Ishikawa

3. Pseudo-contractive operators

The interest in **pseudocontractive** mappings is due mainly to

a) their usefulness as an additional assumption to Lipschitz type conditions in proving convergence of fixed point iterative procedures;

b) their connection with the important class of nonlinear **accretive operators.**

Let H be a Hilbert space and $T : H \to H$ a selfmap.

Definition 1. T is said to be **pseudocontractive** on $C \subset H$ if

$$\|Tx - Ty\| \leq \|x - y\|^2 + \|Tx - Ty - (x - y)\|^2, \quad x, y \in C \tag{10}$$

or equivalently

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \qquad x, y \in C. \tag{10'}$$

Definition 2. T is called **generalized pseudocontractive,** if $\exists r > 0$ such that

$$\|Tx - Ty\|^2 \leq r \cdot \|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2, \quad x, y \in C, \tag{11}$$

or equivalently

$$\langle Tx - Ty, x - y\rangle \leq r \cdot \|x - y\|^2, \qquad x, y \in C \tag{11'}$$

Remark. For $r = 1$, from Definition 2 we obtain Definition 1.

Definition 3. The operator T is said to be **strictly pseudocontractive on C** if there exists $k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\,\|Tx - Ty - (x - y)\|^2, \quad x, y \in C, \tag{12}$$

Defintion 4. The operator T is called **strongly pseudocontractive on C** if there exist $t > 1$ and $r > 0$ such that

$$\|x - y\|^2 \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|, \quad x, y \in C.$$

Remarks. 1) For $t = 1$, from Definition 4 we obtain Definition 1;

2) T is pseudo-conttractive if and only if $I - T$ is accretive ([21]).

The class of pseudo-contractive operators usually associated with lipschitzian properties has been studied extensively by various authors, see [22] and references therein.

The following result has been proved by Verma [12].

THEOREM 4. *Let K be a non-empty closed convex subset of H and $T : K \to K$ a lipschitzian and generalized pseudocontractive operator (with constant s and r, respectively, $r < 1$).*

Then, for any $\lambda, 0 < \lambda < \frac{2(1-r)}{1-2r+s^2}$ the iteration (x_n),given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \qquad n \geq 0$$

converges strongly to the unique fixed point x^* of T :

$$x_n \to x^*(n \to \infty).$$

In our paper [1] we completed the result of Verma by inserting both *a priori* and *a posteriori* error estimates (Theorem 3.1) and by finding the fastest iteration amongst all Krasnoleski-Schafer iterations.

Moreover, when the Picard iteration and Krasnoselski-Schaefer iteration converge simultaneously, it is possible to compare the fastest one, in a certain sense of rate of convergence. Generally, it is possible to find the fastest Krasnoselski-Schaefer iteration in the family, as shown by

THEOREM 5 ([1]). *Let K be a non-empty closed convex subset of a Hilbert space and $T : K \to K$ a generalized pseudocontractive (with constant r, $0 < r < 1$) and lipschitzian (with constant $s > 0$) operator and $\lambda \in (0,1)$ such that $0 < \lambda < \frac{2(1-r)}{1-2r+s^2}$. Then*

(i) T has an unique fixed point $x^* \in K$;

(ii) The Krasnoselski-Schaefer iteration (x_n) converges strongly to x^*, for each $x_0 \in K$;

(iii) $\|x_n - x^*\| \leq \frac{\theta^n}{1-\theta} \cdot \|x_1 - x_0\|, \quad n \geq 1$

$\|x_n - x^*\| \leq \frac{\theta}{1-\theta} \cdot \|x_n - x_{n-1}\|, \quad n \geq 1$

where $\theta = \sqrt{(1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2}$.

(iv) The fastest Krasnoselski-Schaefer iteration is obtained for

$$\theta_0 = \frac{1-r}{1-2r+s^2}.$$

EXAMPLE 3.([1]) Let $K = \left[\frac{1}{2}, 2\right]$ and $T : K \to K$ the function defined by $T(x) = \frac{1}{x}, x \in \left[\frac{1}{2}, 2\right]$ .Then

1) T is Lipschitzian with constant $s = 4$;

2) T is generalized pseudocontractive with constant $r > 0$ arbitrary;

3) The Picard iteration, with $x_0 = a \neq 1$yields the oscillatory sequence

$$a, \frac{1}{a}, \ a, \frac{1}{a}, ...$$

4) The Krasnoselki-Schaefer iteration converges to $x^* = 1$, for any $\lambda \in \left(0, \frac{2(1-r)}{17-2r}\right), r < 1$;

5) For $r = 0,5$, the fastest Krasnoselski-Schaefer iteration is obtained taking $\lambda_0 = \frac{1}{32}$,i.e.

$$x_{n+1} = \frac{1}{32}\left(31x_n + \frac{1}{x_n}\right), \quad n \geq 0$$

which converges to $x^* = 1$, (slowly, because the contraction coefficient $\theta_0 = \frac{\sqrt{63}}{8} \approx 0,992$ is very close to 1).

Final remarks

The class of pseudo-contractive operators has been intensively studied in the last decade (see, for example, [4], [5]). We considered a more general class of pseudo-contractive operators in [20].

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