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FEEDBACK STABILIZATION OF NAVIER–STOKES EQUATIONS

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Abstract. The study of the local exponential stabilization problem for the Navier-Stokes equations using the algebraic Riccati equation is the main aim of this paper.
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1. Introduction

Consider the Navier–Stokes system

$$\begin{aligned} & y_t(x,t) - \nu\Delta y(x,t) + (y\cdot\nabla)y(x,t) = m(x)u(x,t) + \\ & \qquad + f_0(x) + \nabla p(x,t), \quad (x,t)\in Q \\ & (\nabla\cdot y)(x,t) = 0, \qquad \forall\,(x,t)\in Q = \Omega\times(0,\infty) \\ & y = 0, \qquad \text{on } \Sigma = \partial\Omega\times(0,\infty) \\ & y(x,0) = y_0(x), \qquad x\in\Omega \end{aligned} \tag{1.1}$$

in a domain Ω of R^d , $d = 2$ or $d = 3$ where $y = (y_1, ..., y_d)$ is the state $u = (u_1, u_2, ..., u_d)$ is the control input,p is the unknown pressure and m is the characteristic function of an open subset ω of Ω. Let

$$H = \{y \in (L^2(\Omega))^d,\ \nabla\cdot y = 0,\ y\cdot n = 0 \text{ on } \partial\Omega\},\ V = \{y\in (H_0^1(\Omega))^d,\ \nabla\cdot y = 0\}$$

and $P : (L^2(\Omega))^d \longrightarrow H$ be the Leray projector.We set

$$b(y,z,w) = \sum_{i,j=1}^d \int_\Omega y_i D_i z_j w_j dx,\ (By,w) = b(y,y,w),\ \forall\, y,w\in V.$$

and rewrite equation (1.1) as

$$\begin{aligned} & \frac{dy}{dt}(t) + \nu Ay(t) + By(t) = P(mu) + Pf_0, \quad t\in[0,\infty) \\ & y(0) = y_0, \qquad t\in(0,\infty) \end{aligned} \tag{1.1$'$}$$

where $A\in L(V,V')$ (the Stokes operator)

$$(Ay,w) = \sum_{i=1}^d \int_\Omega \nabla y_i\cdot\nabla w_i dx,\ \forall\, y,w\in V.$$

Let (y_e, p_e) be a steady–state solution to (1.1), i.e.,

$$\begin{aligned} &-\nu\Delta y_e + y_e\cdot\nabla y_e = \nabla p_e + f_0(x) && \text{in } \Omega\\ &\nabla\cdot y_e = 0 && \text{in } \Omega\\ &y_e = 0 && \text{on } \partial\Omega. \end{aligned}$$

We assume that *the boundary $\partial\Omega$ is a finite union of $d-1$ dimensional C^∞–connected manifolds diffeomorphic with $S^r_d = \{x\in R^d,\ |x| = r\}$.*

For $d = 2, 3$ always there is at least one steady–state solution but for small viscosity constant ν this solution might be instable. However, by a recent result of O.Imanuvilov [6], [7] (see also [2]) if (y_e, p_e) and y_0 are sufficiently smooth, for instance if

$$(y_e, p_e) \in ((H^3(\Omega))^d)\cap V)\times H^1(\Omega),\ y_0 \in (H^2(\Omega))^d\cap V \tag{1.2}$$

and if $\|y_0 - y_e\|_{(H^2(\Omega))^d} \le \eta$ is sufficiently small then for each $T > 0$ there are

$$u \in H^1(0,T;(L^2(\Omega))^d),\ y \in L^\infty(0,T;(H^2(\Omega))^d\cap V)\cap H^1(0,T;H) \tag{1.3}$$

and $p \in L^2(0,T;H^1(\Omega))$ satisfying (1.1) and such that $y(x,T)\equiv y_e(x)$. In particular, this implies that there is a controller u which stabilizes the steady–state solution y_e. Here we shall treat the local exponential stabilization problem for the Navier–Stokes using the algebraic Riccati equation associated with the linearized Stokes equation.T he argument will be scketched only; the complete proofs will appear elsewhere. For recent results on stabilization of fluid flows we refer to the works [1],[3],[4].

Here $H^k(\Omega)$ and $H^1(0,T;X)$, (X is a Hilbert space) are usual Sobolev spaces. We denote by $|\cdot|$ the norm of H and $(L^2(\Omega))^d$ and by $\|\cdot\|$ the norm of V. By $(\cdot,\cdot)$ denote the pairing between V, V' (the dual space of V) and, respectively, the scalar product of H. Finally, $|\cdot|_s$ is the norm of the Sobolev space $(H^s(\Omega))^d$.

2. Stabilization of the linearized equation

Substituting y by $y + y_e$ into (1.1) we are lead to the equation

$$\begin{aligned} &y_t - \nu\Delta y + (y\cdot\nabla)y + (y_e\cdot\nabla)y + (y\cdot\nabla)y_e = mu + \nabla p && \text{in } Q\\ &\nabla\cdot y = 0 && \text{in } Q\\ &y = 0 && \text{on } \Sigma\\ &y(x,0) = y_0(x) - y_e(x) = y^0(x), && x\in\Omega. \end{aligned} \tag{2.1}$$

Equivalently,

$$\begin{aligned} &\frac{dy}{dt}(t) + \nu Ay(t) + A_0y(t) + By(t) = P(mu),\ t\ge 0\\ &y(0) = y_0 \end{aligned} \tag{2.2}$$

where $A_0 \in L(V,H)$ is defined by

$$(A_0y, w) = b(y_e, y, w) + b(y, y_e, w),\ \forall\, w\in H. \tag{2.3}$$

Consider the linear system

$$\frac{dy}{dt}(t) + \nu Ay(t) + A_0y(t) = P(mu)(t),\ t\ge 0,\ y(0) = y^0 \tag{2.4}$$

and the optimal control problem

$$\varphi(y^0) = \mathrm{Min}\left\{\frac{1}{2}\int_0^\infty (\left|A^{\frac{7}{8}}y(t)\right|^2 + |u(t)|^2)dt;\ (y,u) \text{ subject to } (2.4)\right\}. \tag{2.5}$$

Denote by $D(\varphi)$ the set of all $y^0 \in H$ for which $\varphi(y^0) < \infty$. Under our assumptions for each $y^0 \in H$ the linear Stokes equation (2.4) is exactly null controllable (see [6], [7].) More precisely, there are $u \in H^1(0,T;(L^2(\Omega))^d)$, $y \in L^2(0,T;(H^2(\Omega))^d \cap V)$ solution to (2.4) such that $y(T) \equiv 0$ and (see Lemma 3.1 in [2])

$$|u|_{H^1(0,T;(L^2(\Omega))^d)} \leq C|y^0|.$$

Moreover, we have

$$\varphi(y^0) \geq C\left|A^{\frac{3}{8}}y^0\right|^2. \tag{2.6}$$

because (see e.g. [5],[8])

$$\left|(A_0 y, A^{\frac{3}{4}}y)\right| \leq \left|b(y,y_e,A^{\frac{3}{4}}y)\right| + \left|b(y_e,y,A^{\frac{3}{4}}y)\right| \leq C\|y\|\left|A^{\frac{3}{4}}y\right|.$$

By (2.6) we see that $D(\varphi) = D(A^{\frac{3}{8}}) = W$. The space W is endowed with the graph norm $|y|_W = \left|A^{\frac{3}{8}}y\right|$. Thus there is a linear self–adjoint operator $R : H \to H$ with the domain $D(R) = D(A^{\frac{3}{4}})$ such that

$$\frac{1}{2}(Ry^0, y^0) = \varphi(y^0),\ \forall\, y^0 \in D(A^{\frac{3}{4}}).$$

Moreover, $R \in L(W, W')$ and the latter extends to all of W.

PROPOSITION 1. *Let $d = 2, 3$. Then the optimal control u^* is expressed as*

$$u^*(t) = -mRy^*(t),\ \forall\, t > 0. \tag{2.7}$$

Moreover, there is $\omega > 0$ such that

$$(Ry, y) \geq \omega\|y\|^2{}_W,\ \forall\, y \in W \tag{2.8}$$

and R is the solution to algebraic Riccati equation

$$(\nu Ay + A_0 y, Ry) + \frac{1}{2}|mRy|^2 = \frac{1}{2}\left|A^{\frac{7}{8}}y\right|^2,\ \forall\, y \in D(A). \tag{2.9}$$

Proof. Estimate (2.8) follows by (2.6). For $T > 0$, let (y^*, u^*) be the solution to optimal control problem

$$\mathrm{Min}\left\{\frac{1}{2}\int_0^T (\left|A^{\frac{7}{8}}y(t)\right|^2 + |u(t)|^2)dt + \varphi(y(T)); (y,u) \text{ subject to (2.4) on } (0,T)\right\}.$$

Thus $u^*(t) = mq^T(t),\ \forall t \in [0,T)$ where $q^T \in L^2(0,T;H) \cap C_w([0,T];V')$ is the solution to adjoint equation

$$\begin{aligned} &\frac{d}{dt}q^T - (\nu A + A_0)^* q^T = A^{\frac{7}{4}}y^*,\ t \in (0,T) \\ &q^T(T) = -Ry^*(T). \end{aligned} \tag{2.10}$$

By (2.10) and the unique continuous property for the Stokes equation it follows that $q^T = q^{T'}$ on $(0,T)$ for $0 < T < T'$. Hence $q^T = q$ is independent of T and so the above equations extend to all of R^+. Moreover,

$$q(t) = -Ry^*(t), \ \forall t \geq 0$$

and so we obtain (2.7) as claimed. Next we have

$$\varphi(y^*(t)) = \frac{1}{2}\int_t^\infty (\left|A^{\frac{7}{8}}y^*\right|^2 + |u^*|^2)ds, \ \forall t \geq 0$$

$$\left(Ry^*(t), \frac{dy^*}{dt}(t)\right) + \frac{1}{2}\left|A^{\frac{7}{8}}y^*(t)\right|^2 + \frac{1}{2}|mRy^*(t)|^2 = 0, \text{ a.e. } t > 0.$$

This yields

$$-(Ry^*(t), \nu Ay^*(t) + A_0y^*(t)) - \frac{1}{2}|mRy^*(t)|^2 + \frac{1}{2}\left|A^{\frac{7}{8}}y^*(t)\right|^2 = 0, \ \forall t \geq 0.$$

3. Stabilization of the Navier–Stokes equation

THEOREM 1. *Let $d = 2, 3$ and let R be the operator defined in Proposition 1. Let (y_e, p_e) $((H^3(\Omega))^d \cap V) \times H^1(\Omega)$ be a steady–state solution to equation (1.1). Then the feedback controller*

$$u = -mR(y - y_e) \tag{3.1}$$

exponentially stabilizes y_e in a neighbourhood $\mathcal{V} = \{y_0 \in W;\ \|y_0 - y_e\|_W < \rho\}$ of y_e. More precisely, for each $y_0 \in \mathcal{V}$ there is a weak solution $y \in L^\infty_{\text{loc}}(R^+; H) \cap L^2_{\text{loc}}(R^+; V)$ to closed loop system

$$\begin{aligned} &\frac{dy}{dt} + \nu Ay + A_0y + By + P(mR(y - y_e)) = Pf_0, \ t > 0 \\ &y(0) = y_0 \end{aligned} \tag{3.2}$$

such that

$$|y(t) - y_e| \leq Ce^{-\gamma t}\|y_0 - y_e\|_W, \ \forall y_0 \in \mathcal{V} \tag{3.3}$$

for some $\gamma > 0$.

Proof. We reduce the problem to that of stability of the null solution to corresponding closed loop system (2.2), i.e.,

$$\frac{dy}{dt} + \nu_0 Ay + A_0y + By + P(mRy) = 0, \ t > 0, \ y(0) = y^0. \tag{3.4}$$

We consider the approximating equation

$$\begin{aligned} &\frac{dy_N}{dt} + \nu_0 Ay_N + A_0y_N + B_Ny_N + P(mRy_N) = 0, \\ &y_N(0) = y^0, \end{aligned} \tag{3.5}$$

where

$$B_Ny = By \text{ if } \|y\| \leq N, \ B_Ny = \frac{N^2}{\|y\|^2}By \text{ if } \|y\| > N.$$

For each $y^0 \in D(A)$ equation (3.5) has a unique solution$y_N \in W^{1,\infty}_{\rm loc}(R^+;H) \cap L^\infty_{\rm loc}(R^+;D(A))$. If $y^0 \in V$ then

$$|y_N(t)|^2 + \int_0^t \|y_N(s)\|^2 ds \le C_T,\ \forall\, t \in (0,T), \tag{3.6}$$

where C_T is independent of N. Thus on a subsequence $N \to \infty$

$$y_N \longrightarrow y \text{ weak star in } L^\infty_{\rm loc}(R^+;H) \cap L^2_{\rm loc}(R^+;V) \tag{3.7}$$

where $y \in L^2_{\rm loc}(R^+;V) \cap C_w(R^+;H)$ is a weak solution to equation (3.4). Now we multiply equation (3.5) by Ry_N and use (2.9) to obtain that

$$\begin{aligned} &\frac{d}{dt}(Ry_N(t), y_N(t)) + |mRy_N(t)|^2 + \left|A^{\frac34} y_N(t)\right|^2 = \\ &= 2(B_N y_N(t), Ry_N(t)),\ \text{a.e. } t > 0. \end{aligned} \tag{3.8}$$

On the other hand, we have by (2.7) and the properties of b (see e.g. [5], [8]) that

$$\begin{aligned} |(B_N y_N, y_N)| &\le \inf\left(1, \frac{N^2}{\|y_N\|^2}\right) |b(y_N, y_N, Ry_N)| \le \\ &\le C|y_N|_{\frac34} |y_N|_{\frac74} |Ry_N| \le C\left|A^{\frac38} y_N\right|\left|A^{\frac78} y_N\right|\left|A^{\frac34} y_N\right| \le \\ &\le C\left|A^{\frac78} y_N\right|^2 (Ry_N, y_N)^{\frac12} \end{aligned} \tag{3.9}$$

because $D(A^m) \subset (H^{2m}(\Omega))^n$ for all noninteger m. We set

$$E = \{y^0 \in W;\ (Ry^0, y^0) < \rho\}.$$

By (3.8) and (3.9) we see that for ρ sufficiently small and $y^0 \in E$ we have

$$\frac{d}{dt}(Ry_N(t), y_N(t)) + \frac12\left|A^{\frac78} y_N(t)\right|^2 \le 0,\ \text{a.e. } t > 0$$

and this yields

$$|y_N(t)| \le \|y_N(t)\|_W \le C\left\|y^0\right\|_W e^{-\gamma t},\ \forall\, t \ge 0$$

for some $\gamma > 0$. Then we see by (3.7) that

$$|y(t)| \le C\left\|y^0\right\|_W e^{-\gamma t}.\ \forall\, t \ge 0$$

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