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# A GENERALIZATION OF MIRANDA'S THEOREM

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**Abstract.** Let $P = \prod_{i=1}^{N} [-1,1] \subset \mathbb{R}^N$ and let $f : P \to \mathbb{R}^N$ be a continuous function. In 1940 C. Miranda proved that if each component $f_i$ of the function $f$ takes values with contrary signs on the faces of the cube $P$ obtained through the intersection of this with the planes $x_i = \pm 1$, then $f$ admits zeros in $P$. Moreover, C. Miranda showed that his result is equivalent with the Brouwer's fixed point theorem. The following question appears naturally: the result of C. Miranda can be extended to infinite dimensional spaces? In this paper a partial answer to this question is given.
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## 1. Introduction

The first general result looking the existence of solutions for the equation

$$f(x) = 0 \tag{1}$$

is the well known Cauchy theorem which ensures that for a given continuous mapping $f : [a,b] \to \mathbb{R}$ the condition $f(a) \cdot f(b) \leq 0$ is sufficient to conclude that the equation (1) admits solutions.

This result has been generalized in 1940 by C. Miranda [1] in the following sense. Let $P = \prod_{i=1}^{N} [-1,1]$ and $f : P \to \mathbb{R}^N$ be a continuous mapping. If each component $f_i$ of the function $f$ satisfies the conditions

$$f_i(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_N) \geq 0,\ \ f_i(x_1, ..., x_{i-1}, -1, x_{i+1}, ..., x_N) \leq 0, \tag{2}$$

for every $i \in \overline{1,N}$ and $|x_j| \leq 1$, for $j \neq i$, then the equation (1) admits solutions in $P$.

Denoting by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^N$, one remarks that the condition (2) can be obtained from

$$\langle x, f(x) \rangle \leq 0,\ x \in S,$$

where $S$ is the boundary of the unity ball in $\mathbb{R}^N$ endowed with the norm $|x| = \max_{1 \leq i \leq N} \{|x_i|\}$, where $x = (x_i)_{1 \leq i \leq N}$.

Now we deal with the natural question if we can extend Miranda's result to infinite dimensional spaces. A partial answer to this question is given in the present paper, generalizing the Miranda's result to the case of continuous functions on a compact spaces. Moreover, we prove that the result is equivalent with the Schauder's fixed point theorem, as the Miranda's theorem is equivalent with the Brouwer's fixed point theorem.

Finally, the last part of this paper containes an example where The Brouwer's theorem is used to proving the existence of solutions for the equation (1) in infinite dimensional spaces.

## 2. A general existence result

Let $H$ be a Banach space endowed with an inner product $\langle \cdot \mid \cdot \rangle$ (i.e. $H$ is a pre-Hilbert space) and let $K$ be an arbitrary topological compact space. Set

$$X = \{x : K \to H,\, x \text{ continuous}\}$$

and define in $X$ the usual norm $\|\cdot\|$ by

$$\|x\| := \sup_{t \in K} \|x(t)\| .$$

As usually we consider

$$\begin{aligned} \overline{B} &= \{x \in X,\ \ \|x\| \le 1\} \\ S &= \{x \in X,\ \ \|x\| = 1\} . \end{aligned}$$

Theorem 1. *Let $f : \overline{B} \to X$ be an operator satisfying the following conditions:*
*i) $f$ is compact;*
*ii) $0 \notin \left(\partial f\left(\overline{B}\right) \setminus f\left(\overline{B}\right)\right)$;*
*iii) $\langle x(t) \mid (f(x))(t) \rangle \le 0$, $(\forall)\ x \in S$, $(\forall)\ t \in K$.*
*Then the equation*

$$f(x) = 0 \tag{3}$$

*admits solutions in $\overline{B}$.*

**Proof.** Suppose that

$$f(x) \neq 0,\ \ (\forall)\ \ x \in \overline{B}.$$

Then, from $0 \notin f\left(\overline{B}\right)$ and hypothesis ii) it results

$$0 \notin \overline{f\left(\overline{B}\right)}$$

and therefore the operator $F : \overline{B} \to \overline{B}$ defined by

$$F(x) := \frac{1}{\|f(x)\|} \cdot f(x)$$

is compact and hence, based on Schauder's theorem, admits fixed points in $\overline{B}$.

If

$$x = F(x)$$

then

$$x \in S \tag{4}$$

and

$$x \cdot \|f(x)\| = f(x) \tag{5}$$

By (5) and hypothesis iii) it results

$$\|f(x)\| \cdot \langle x(t) \mid x(t)\rangle = \langle x(t) \mid (fx)(t)\rangle \leq 0, \ \ (\forall)\ \ t \in K. \tag{6}$$

So,

$$\langle x(t) \mid x(t)\rangle \leq 0, \ \ (\forall)\ \ t \in K$$

allowing us to conclude that

$$x(t) \equiv 0, \tag{7}$$

which contradicts (4). The theorem in proved.

## 3. Equivalence result

One consider the natural problem of knowing if the existence results of type proved above are equivalent with Schauder's fixed point theorem, as it happens with Miranda's theorem and Brouwer's theorem.

A partial answer is given in this section in the particular case $H = \mathbb{R}^N$.

Consider $X = C\left(K, \mathbb{R}^N\right)$ and in $\mathbb{R}^N$ the norm

$$|x| := \max_{1 \leq i \leq N} \{|x_i|\}.$$

Keeping the notations from the previous section set

$$\begin{aligned} S_i^+ \ &:= \left\{x = (x_j)_{j \in \overline{1,N}} \in \overline{B},\ x_i \equiv 1\right\} \\ S_i^- \ &:= \left\{x = (x_j)_{j \in \overline{1,N}} \in \overline{B},\ x_i \equiv -1\right\}. \end{aligned}$$

Let $f : \overline{B} \to X$ be a compact operator; consider $f(x) = (f_i(x))_{i \in \overline{1,N}}$ to praise the components of function $f(x) : K \to \mathbb{R}^N$.

Definition 1. *We say that the operator $f : \overline{B} \to X$ is of class $(M)$ on $X$ iff:*

*a)* $0 \notin \left(\partial f\left(\overline{B}\right) \setminus f\left(\overline{B}\right)\right)$;

*b)* $(f_i(x))(t) \leq 0$, *for every* $x_i \in S_i^+$, $t \in K$, $i \in \overline{1,N}$ *and* $(f_i(x))(t) \geq 0$, *for every* $x_i \in S_i$, $t \in K$, $i \in \overline{1,N}$.

The symbol "$\partial$" denotes as usually the boundary operator in $X$.

Let $Y$ be a (closed) subspace of $X$.

Definition 2. *We say that the operator $f$ is of class $(M)$ on $Y$ iff $f\left(\overline{B}\right) \subset Y$ and the conditions a), b) are satisfied on $\overline{B} \cap Y$.*

Definition 3. *We say that the operator $f : \overline{B} \to X$ is of class $(S)$ iff $f\left(\overline{B}\right) \subset \overline{B}$.*

Theorem 2. *If every operator of class $(S)$ admits fixed points, then every operator of class $(M)$ on $X$ admits zeros in $\overline{B}$.*

*Conversely, if every operator of class $(M)$ on an arbitrary subspace $Y$ admits zeros, then every operator of class $(S)$ admits fixed points.*

**Proof.** Firstly, let us suppose by mean of contradiction that

$$f(x) \neq 0, \quad (\forall) \ x \in \overline{B}. \tag{8}$$

Define as above the operator $F : \overline{B} \to \overline{B}$ by

$$F(x) = \frac{1}{\|f(x)\|} \cdot f(x)$$

The operator $F$ being of class $(S)$ it admits fixed points in $\overline{B}$. If $x = (x_i)_{i \in \overline{1,N}}$ is a fixed point, then it satisfies the conditions (4) and (5). One obtains by (4)

$$(\exists) \ j \in \overline{1,N}, \quad (\exists) \ t_0 \in K, \quad |x_j(t_0)| = 1 \tag{9}$$

and by (5)

$$x_i(t) \cdot \|f(x)\| = (f_i(x))(t), \quad (\forall) \ i \in \overline{1,N}, \quad (\forall) \ t \in K. \tag{10}$$

Suppose firstly that

$$x_j(t_0) = 1. \tag{11}$$

Then, by (10) we get

$$(f_j(x))(t_0) > 0. \tag{12}$$

By other part, setting $\widetilde{x} := (x_1, ..., x_{j-1}, 1, x_{j+1}, ..., x_N)$ we have obviously $\widetilde{x} \in S_j^+$ and so

$$(f_j(\widetilde{x}))(t) \leq 0, \quad (\forall) \ t \in K. \tag{13}$$

Since $x(t_0) = \widetilde{x}(t_0)$, the relations (12) and (13) follow us to a contradiction.

One treat similarly the case

$$x_j(t_0) = -1.$$

Let us prove the second part of theorem. So, let $f : \overline{B} \to \overline{B}$ be a compact operator. Suppose that

$$f(x) \neq x, \quad (\forall) \ x \in \overline{B}. \tag{14}$$

Based on a known property of compact operators, by (14) it results that there exists $a > 0$ such that

$$\|f(x) - x\| > 2a, \quad (\forall) \ x \in \overline{B}. \tag{15}$$

If

$$\alpha \in (1, 1 + a), \tag{16}$$

we have

$$\|f(x) - \alpha x\| \geq \|f(x) - x\| - (\alpha - 1) \cdot \|x\| > 2a - 1 + \alpha > a. \tag{17}$$

Bu using the well-known Schauder's theorem, for every $\varepsilon > 0$ there exists an operator $f^\varepsilon : \overline{B} \to Y_\varepsilon \subset X$ such that

$$\|f^\varepsilon(x) - f(x)\| < \varepsilon, \quad (\forall) \ x \in \overline{B}, \tag{18}$$

$$\dim Y_\varepsilon < \infty.$$

Define an operator $\Phi^\varepsilon : \overline{B} \cap Y_\varepsilon \to Y_\varepsilon$ by

$$\Phi^\varepsilon(x) := f^\varepsilon(x) - \alpha x.$$

Clearly, $\Phi^\varepsilon$ is a compact operator. Denote by $\Phi_i^\varepsilon$, $i \in \overline{1,N}$ the components of $\Phi^\varepsilon$. If $x \in S_i^+ \cap Y_\varepsilon$ and $\varepsilon < \alpha - 1$, we have

$$(19) \qquad (\Phi_i^\varepsilon(x))(t) = (f_i^\varepsilon(x))(t) - (f_i(x))(t) + (f_i(x))(t) - \alpha x_i < \varepsilon - \alpha + 1 < 0.$$

Similarly, if $x \in S_i^- \cap Y_\varepsilon$, we have

$$(20) \qquad (\Phi_i^\varepsilon(x))(t) > -\varepsilon - 1 + \alpha > 0.$$

The inequalities (19) and (20) holds if

$$(21) \qquad 0 < \varepsilon < \alpha - 1.$$

On the other hand,

$$(22) \qquad \begin{aligned} \|\Phi^\varepsilon(x)\| &= \|f^\varepsilon(x) - \alpha x\| \geq \|f(x) - \alpha x\| - \|f^\varepsilon(x) - f(x)\| > \\ &> a - \varepsilon > 0, \end{aligned}$$

because, from (21) it results that $\varepsilon < a$.

From (22) one obtains that

$$(23) \qquad 0 \notin \Phi^\varepsilon\left(\overline{B} \cap Y_\varepsilon\right).$$

But $\Phi^\varepsilon\left(\overline{B} \cap Y_\varepsilon\right)$ is a closed set and hence from (22) it follows

$$(24) \qquad 0 \notin \partial\Phi^\varepsilon\left(\overline{B} \cap Y_\varepsilon\right).$$

Consequently, $\Phi^\varepsilon$ is an operator of class $(M)$ on $Y_\varepsilon$ and therefore

$$(25) \qquad 0 \in \Phi^\varepsilon\left(\overline{B} \cap Y_\varepsilon\right).$$

But the relations (22) and (24) being in contradiction, the relation (14) is not true.

We present a corollary of Theorem 2 which can be used as an existence theorem.

Corollary 1. *Let $f : \overline{B} \to X$ be a compact operator. If the conditions a) and b) of Definition 1 are fulfilled, then $f$ admits zeros in $\overline{B}$.*

## 4. Existence result for periodic solutions

Let $f : [0, \omega] \to \mathbb{R}^N$ be a continuous function. Consider the problem

$$(26) \qquad \dot{x} = f(t, x)$$

$$(27) \qquad x(0) = x(\omega).$$

Clearly, a solution $x = x(t)$ of the equation (26) satisfies (27) if and only if

$$(28) \qquad \int_0^\omega f(s, x(s))\ ds = 0.$$

More precisely, the problem $(26), (27)$ admits solutions if and only if the integral equation (28) admits solutions in the set of solutions of the eqaution $(26)$.

Consider $x = (x_i)_{i\in\overline{1,n}} \in \mathbb{R}^N$. We put $\|x\| := \max\left\{|x_i|\,,\, i\in\overline{1,N}\right\}$; similarly, if $f : [0,\omega]\times\mathbb{R}^N \to \mathbb{R}^N$ is a continuous function we put $f = (f_i)_{i\in\overline{1,N}}$. Consider now the problem

$$\dot{x} = f(t,x)\,, \tag{29}$$

$$x(0) = x(\omega)\,. \tag{30}$$

Theorem 3. *Assume that the following hypotheses are fulfilled:*

*i)* $f : [0,\omega]\times\mathbb{R}^N \to \mathbb{R}^N$ *is continuous and bounded;*

*ii) the Cauchy problem*

$$\begin{cases} \dot{x} = f(t,x) \\ x(0) = c \end{cases} \tag{31}$$

*admits a unique solution, for all* $c\in\mathbb{R}^N$, $c = (c_i)_{i\in\overline{1,N}}$;

*iii) there exists a constant* $\alpha > 0$ *such that for every* $i\in\overline{1,N}$

$$x_i\cdot f_i(t,x) > 0,\ (\forall)\ t\in[0,\omega]\,,\ (\forall)\ x\in\mathbb{R}^N,\ |x_i| > \alpha. \tag{32}$$

*Then the problem* $(29), (30)$ *admits solutions.*

Proof. Denote by $x(t;c)$ the unique solution of the problem $(31)$. Since is well known, the boundedness of $f$ assures the existence of $x(t;c)$ on the whole interval $[0,\omega]$. Simultaneously, the unicity of solution for the problem (31) attracts the continuity of the map $c\to x(\cdot;c)$ considered on from $\mathbb{R}^N$ to $C\left([0,\omega]\,,\,\mathbb{R}^N\right)$.

Let

$$M = \sup\left\{\|f(t,x)\|\,,\,(t,x)\in[0,\omega]\times\mathbb{R}^N\right\}.$$

Since

$$x_i(t;c) = c_i + \int_0^t f_i(s, x(s;c))\,ds \tag{33}$$

we obtain

$$c_i - M\omega \le x_i(t;c) \le c_i + M\omega.$$

Hence, for every $i\in\overline{1,N}$ we have

$$x_i(t;c) > \alpha,\ \text{ if } c_i > \alpha + M\omega \tag{34}$$

and

$$x_i(t;c) < -\alpha,\ \text{ if } c_i < -\alpha - M\omega, \tag{35}$$

for every $t\in[0,\omega]$ and $c\in\mathbb{R}^N$. By $(34), (35)$ and hypothesis *iii)* there results that for all $i\in\overline{1,N}$, $t\in[0,\omega]$ and $c\in\mathbb{R}^N$ we have

$$f_i(t, x(t;c)) > 0,\ \text{ if } c_i > \alpha + M\omega \tag{36}$$

and

$$f_i(t, x(t;c)) < 0,\ \text{ if } c_i < -\alpha - M\omega. \tag{37}$$

Consider $l > \alpha + M\omega$, let

$$K := \left\{ c \in \mathbb{R}^N,\ |c_i| \leq l, \text{ for all } i \in \overline{1,N} \right\} \tag{38}$$

and define on $K$ the operator

$$F(c) := \int_0^\omega f(s, x(s;c))\, ds.$$

Obviously, $F : K \to \mathbb{R}^N$ is a continuous operator and $K$ is a convex and compact set.

We want to show that

$$\text{there exists } c \in K \text{ such that } F(c) = 0. \tag{39}$$

Indeed, let us suppose that (39) is not true; then we define the operator $\Phi : K \to K$ by

$$\Phi(c) := -l \frac{F(c)}{\|F(c)\|},\ \Phi = (\Phi_i)_{i \in \overline{1,N}}. \tag{40}$$

From the Brouwer's theorem $\Phi$ admits fixed points; therefore

$$\text{there exists } c = (c_i)_{i \in \overline{1,N}} \in K \text{ such that } c = \Phi(c). \tag{41}$$

By (40) it follows that $\|c\| = l$; but from $|c_i| \leq l$ it results that there exists $i \in \overline{1,N}$ such that $|c_i| = l.$ If $c_i = l$, from (39) it follows that $F_i(c) < 0$ and if $c_i = -l$ we obtain $F_i(c) > 0$, which are in contradiction with (36) and respectively (37).

So, consider $c \in K$ such that $F(c) = 0$; since

$$x(\omega; c) = c + \int_0^\omega f(s, x(s;c))\, ds$$

it follows that

$$x(0;c) = x(\omega;c).$$

The proof is now complete.

## References

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