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ON A k-COMPLEX MOMENT PROBLEM

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Abstract. In this paper we give a necessary and sufficient condition on a sequence $\{\Gamma_{\alpha,\beta} = (s_{ij}(\alpha,\beta))_{1\le i,j\le k}, \alpha,\beta \in \mathbb{N}^n\}_{\alpha,\beta}$ of (k,k) matrices with complex entries, $k \in \mathbb{N}^*$, to be a complex moment sequence with respect to a (k,k) positive defined matrix of Borel measures on the unit polydisc. The proof in this note is different from the proof of a similar result in [Theorem 1.4.8, 19] in case that $\Gamma_{\alpha,\beta}$ are bounded operators acting on an arbitrary Hilbert space, with $\Gamma_{\alpha,\beta} = \Gamma^*_{\beta,\alpha}$. The proof in this note also omits the condition $\Gamma_{\alpha,\beta} = \Gamma^*_{\beta,\alpha}$ on the sequence of matrices $\{\Gamma_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$ from the hypothesis of [Theorem 1.4.8, 19].

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1. INTRODUCTION

Let $\{\Gamma_{\alpha,\beta} = (s_{ij}(\alpha,\beta)_{1\le i,j\le k} \in M(k,\mathbb{C})\}_{\alpha,\beta\in\mathbb{N}^n,k\in\mathbb{N}^*}$ be a multisequence of k-dimensional matrices with complex entries. In this note, we give a necessary and sufficient condition for the existence of a positive defined matrix of complex Borel measures $\Lambda = (\lambda_{ij})_{1\le i,j\le k}$ defined on the closed unit polydisc $D_1^n = \{z = (z_1,...,z_n), |z_i| \le 1, \forall 1 \le i \le n\}$, such that we have the representations: $\Gamma_{\alpha,\beta} = (\int_{D_1^n} z^\alpha \overline{z^\beta} \mathrm{d}\lambda_{ij}(z))_{1\le i,j\le k} \overset{not}{=} \int_{D_1^n} z^\alpha \overline{z^\beta} \mathrm{d}\Lambda(z)$ for all $\alpha,\beta \in \mathbb{N}^n$. The problem formulated above will be called the *k-dimensional complex moment problem.* A different solution of this problem in the case that $\{\Gamma_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$ is a sequence of bounded operators acting on an arbitrary complex Hilbert space $\mathbf{H}$ with $\Gamma_{\alpha,\beta} = \Gamma^*_{\beta,\alpha}$ for all $\alpha,\beta \in \mathbb{N}^n$ was given in [Theorem 1.4.8, 19]. Sections 1 and 2 contain some preliminaries, definitions and notations needed in this note. In Section 3 we give a necessary and sufficient condition for the existence of a solution of the k-complex moment problem.

2. THE k-COMPLEX MOMENT PROBLEM

Let $z = (z_1,...,z_n)$ denote the complex variable in $\mathbb{C}^n$ and D_1^n the closed n-dimensional unit polydisc; for $\alpha = (\alpha_1,...,\alpha_n)$, $\beta = (\beta_1,\ldots,\beta_n) \in \mathbb{N}^n$, we denote with $z^\alpha = z_1^{\alpha_1}\ldots z_n^{\alpha_n}$, $\overline{z}^\beta = \overline{z_1}^{\beta_1}\ldots\overline{z_n}^{\beta_n}$.

Definition 2.1. A k-dimensional matrix $\Lambda = (\lambda_{ij})_{1\le i,j\le k}$ of complex measures is *positive defined* on D_1^n if the following conditions hold:

(a) $\Lambda(M) = (\lambda_{ij}(M))_{1\le i,j\le k}$ is a nonnegative matrix for each Borel set $M \in Bor(D_1^n)$,

(b) for all $1 \leq i, j \leq k$, the positive Borel measures $|\lambda_{ij}|$ on D_1^n have complex moments of all orders.

Definition 2.2. The multisequence of k-dimensional matrices $\{\Gamma_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$ is called a *k-complex moment sequence* if there exists a k-dimensional matrix of complex measures $\Lambda = (\lambda_{ij})_{1\leq i,j\leq k}$, positive defined on D_1^n, such that

$$\Gamma_{\alpha,\beta} = (s_{ij}(\alpha,\beta))_{1\leq i,j\leq k} = \left(\int_{D_1^n} z^\alpha \overline{z}^\beta \mathrm{d}\lambda_{ij}(z)\right)_{1\leq i,j\leq k} = \int_{D_1^n} z^\alpha \overline{z}^\beta \mathrm{d}\Lambda(z)$$

for all $\alpha, \beta \in \mathbb{N}^n$.

Let $\wp$ be the $\mathbb{C}$-vector space of polynomials in $z, \overline{z}$ with complex coefficients and the $\mathbb{C}$-linear mapping L associated with $\{\Gamma_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$, $L : \wp \to M(k,\mathbb{C})$, $L(p) = (l_{ij}(p))_{1\leq i.j\leq k}$, defined by $L(p) = \sum_{\alpha,\beta\in H} a_{\alpha\beta}\Gamma_{\alpha,\beta}$ with $p(z,\overline{z}) = \sum_{\alpha,\beta\in H} a_{\alpha\beta} z^\alpha \overline{z^\beta}$, where $H \subset \mathbb{N}^n$ is finite.

Definition 2.3. The linear mapping $L(.) = (l_{ij}(.))_{1\leq i,j\leq k}$ from $\wp$ into $M(k,\mathbb{C})$ is called *positive* on the compact D_1^n if $\sum_{1\leq i,j\leq k} l_{ij}(p) t_i \overline{t_j} \geq 0$, for all elements $t_i, t_j \in \mathbb{C}$, and all polynomials $p \in \wp$ with $p(z,\overline{z}) \geq 0$, for all $z \in D_1^n$.

3. EXISTENCE OF A SOLUTION

Proposition 3.1. *Let $\{\Gamma_{\alpha,\beta} = (s_{ij}(\alpha,\beta))_{1\leq i,j\leq k} \in M(k,\mathbb{C})\}_{\alpha,\beta\in\mathbb{N}^n}$ be a multisequence of k-dimensional matrices and $L(.) = (l_{ij}(.))_{1\leq i,j\leq k}$ be the associated linear mapping from $\wp$ into $M(k,\mathbb{C})$. Then the following assertions are equivalent*:

(i) *L is positive defined on the compact D_1^n.*

(ii) *$\{\Gamma_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$ is a k-complex moment sequence on D_1^n.*

Proof. (i)⇒ (ii) Let $p \in \wp$, $p(z,\overline{z}) = \sum_{\alpha,\beta\in H\subset\mathbb{N}^n} a_{\alpha\beta} z^\alpha \overline{z}^\beta$, H a finite set in $\mathbb{N}^n$ with $p(z,\overline{z}) \geq 0$ for all $z \in D_1^n$. By (i), we have that $\sum_{1\leq i,j\leq k} l_{ij}(p) t_i \overline{t_j} \geq 0$ for any $t_i, t_j \in \mathbb{C}$. It follows that $l_{ii}(p) \geq 0$, for any $1 \leq i \leq k$, and $\sum_{i=1}^k l_{ii}(p) \geq 0$. Put in the previous inequality $t_i = x,\ x \in \mathbb{R},\ t_j = 1$, and $t_r = 0$ for any $r \in \overline{1,k}$, $r \neq i, j$. In this case we obtain:

$$l_{ii}(p)x^2 + [l_{ij}(p) + l_{ji}(p)]x + l_{jj}(p) \geq 0\,, \forall x \in \mathbb{R}. \tag{1}$$

Inequality (1) implies

$$\mathrm{Im}[l_{ij}(p)] = -\mathrm{Im}[l_{ji}(p)] \tag{1'}$$

and

$$[\mathrm{Re}(l_{ij}(p) + l_{ji}(p))]^2 \leq 4 l_{ii}(p) l_{jj}(p), \text{ for all } p \in \wp \text{ with } p(z,\overline{z}) \geq 0. \tag{1''}$$

If we take also $t_i = x,\ x \in \mathbb{R},\ t_j = \mathrm{i}$, and $t_r = 0$ for any $r \in \overline{1,k}, r \neq i, j$, we obtain

$$l_{ii}(p)x^2 + \mathrm{i}x[l_{ji}(p) - l_{ij}(p)] + l_{jj}(p) \geq 0. \tag{2}$$

From this inequality we have

$$\mathrm{Re}(l_{ij}(p)) = \mathrm{Re}(l_{ji}(p)) \tag{2'}$$

and

$$[\mathrm{Im}(l_{ji}(p) - l_{ij}(p))]^2 \le 4l_{ii}(p)l_{jj}(p). \tag{2''}$$

From (1′), (1″), (2′), and (2″) we get

$$|\mathrm{Re}(l_{ij}(p))|^2 \le l_{ii}(p)l_{jj}(p) \tag{3}$$

and

$$|\mathrm{Im}(l_{ij}(p))|^2 \le l_{ii}(p)l_{jj}(p), \tag{3'}$$

inequalities that are true for all $p \in \wp$ with $p(z,\overline{z}) \ge 0$, when $z \in D_1^n$. Consequently, from (3) and (3′) we get

$$|l_{ij}(p)| \le 2l_{ii}^{\frac{1}{2}}(p)l_{jj}^{\frac{1}{2}}(p) \le l_{ii}(p) + l_{jj}(p) \le \sum_{i=1}^{k} l_{ii}(p), \tag{4}$$

for all $i,j \in \overline{1,k}$, and for all $p \in \wp$ with $p(z,\overline{z}) \ge 0$, when $z \in D_1^n$.

Let

$$\wp_{an} = \left\{ p(z) = \sum_{\alpha \in H \subset \mathbb{N}^n} a_\alpha z^\alpha,\ H \text{ finite and } a_\alpha \in \mathbb{C} \right\}$$

be the analytical polynomials. For any $p \in \wp_{an}$, we define

$$\widetilde{p}(z,\overline{z}) = p(z)\overline{p(z)} = |p(z)|^2.$$

From the previous assertions and notations we have $l_{ii}(\widetilde{p}) \ge 0$, for all $1 \le i \le k$, and $l_{ij}(z^\alpha \overline{z}^\beta) = s_{ij}(\alpha,\beta)$, for any $\alpha,\beta \in \mathbb{N}^n$. If we consider $p(z) = \sum_{\alpha \in H \subset \mathbb{N}^n} a_\alpha z^\alpha$, H finite, and $\widetilde{p}(z,\overline{z}) = |p(z)|^2 = \sum_{\alpha,\beta \in H \subset \mathbb{N}^n} a_\alpha \overline{a_\beta} z^\alpha \overline{z^\beta}$, we obtain

$$0 \le l_{ii}(\widetilde{p}(z,\overline{z})) = \sum_{\alpha,\beta \in H \subset \mathbb{N}^n} a_\alpha \overline{a_\beta} l_{ii}(z^\alpha \overline{z^\beta}) = \sum_{\alpha,\beta \in H \subset \mathbb{N}^n} a_\alpha \overline{a_\beta} s_{ii}(\alpha,\beta).$$

We consider on the $\mathbb{C}$-vector space $\wp_{an}$ the inner products:

$$< p, q >_{s_{ii}} = \sum_{p,q \in H \subset \mathbb{N}^n} a_p \overline{b_q} s_{ii}(p,q),\ \forall 1 \le i \le k,$$

when $p(z) = \sum_{p \in H_1 \subset \mathbb{N}^n} a_p z^p$ and $q(z) = \sum_{q \in H_2 \subset \mathbb{N}^n} b_q z^q$ with H_1, H_2 finite subsets in $\mathbb{N}^n$. With these inner products we organize $\wp_{an}$ as pre-Hilbert spaces. Let $\mathbf{H_i}$ be the Hilbert spaces obtained as the separate completions of $\wp_{an}$ with respect to these inner products for all $1 \le i \le k$. We define on the pre-Hilbert spaces $(\wp_{an}, <,>_{s_{ii}})$ the operators $S_j^i : (\wp_{an} <,>_{s_{ii}}) \to (\wp_{an}, <,>_{s_{ii}})$, $S_j^i p = z_j p$, for all $1 \le j \le n$ and all $1 \le i \le k$. Since for any $p \in \wp_{an}$ and any $z \in D_1^n$ we have $|p(z)|^2(1 - |z_j|^2) \ge 0$, the inequalities $l_{ii}(|p(z)|^2) \ge l_{ii}(|z_j|^2 |p(z)|^2)$, $\forall 1 \le i \le k$, are also true. That means that all operators S_j^i are contractions on $(\wp_{an}, <,>_{s_{ii}})$ when $1 \le i,j \le k$. We denote the extensions of S_j^i to $\mathbf{H_i}$ also with S_j^i, for all $1 \le j \le n$; for the extended operators , we have also $||S_j^i||_{\mathbf{H_i}} \le 1$. The bounded, commuting multioperator $(S_1^i, \ldots, S_n^i) \in L(\mathbf{H_i})^n$ verifies Ito's necessary and sufficient condition to be a subnormal-tuple, that is:

$$\sum_{I=(i_1,\ldots,i_n),J=(j_1,\ldots,j_n)} < (S_1^i)^{i_1} \ldots (S_n^i)^{i_n} p_J, (S_1^i)^{j_1} \ldots (S_n^i)^{j_n} p_I >_{s_{ii}} \ge 0$$

$$\Leftrightarrow l_{ii} \left(| \sum_{J=(j_1,\ldots,j_n) \in H} \overline{z_1^{j_1} \ldots z_n^{j_n}} p_J(z)|^2 \right) \ge 0,$$

for all finite sets of polynomials $\{p_J\}_{J \in H}$, H finite, $p_J \in \wp_{an}$. Hence, from Theorem 1 in [5], we know that there exist Hilbert spaces $\mathrm{K}_i \subset \mathbf{H_i}$, $1 \le i \le k$, and normals $N_j^i : \mathrm{K}_i \to \mathrm{K}_i$, for all $1 \le j \le n$, such that $N_j^i(\mathbf{H_i}) \subset \mathbf{H_i}$, $N_j^i|_{\mathbf{H_i}} = S_j^i$, and $N_j^i = \int_{D_1^n} z_j \mathrm{d}E^i(z_1, ..., z_n)$, with E^i the joint spectral measure of the bounded and commuting multioperator $(N_1^i, \ldots, N_n^i)$, $1 \le i \le k$. Let $[l_0]_i$ be the unit element of $\wp_{an} \subset \mathbf{H_i}$. We define, for any $B \in Bor(D_1^n)$, the positive scalar measures

$$\lambda_i(B) =< E^i(B)[l_0]_i, [l_0]_i >_{s_{ii}}, \ 1 \le i \le k.$$

We get the following representations with respect to these measures:

$$\begin{aligned} & l_{ii}(z^\alpha \overline{z}^\beta) = s_{ii}(\alpha, \beta) \\ & =< (S_1^i)^{\alpha_1} \ldots (S_n^i)^{\alpha_n} [l_0]_i, (S_1^i)^{\beta_1} \ldots (S_n^i)^{\beta_n} [l_0]_i >_{s_{ii}} \\ & = \int_{D_1^n} z^\alpha \overline{z}^\beta \mathrm{d}\lambda_i(z), \ \forall \alpha = (\alpha_1, \ldots, \alpha_n), \ \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n, \ \forall 1 \le i \le k. \end{aligned}$$

More details about this construction are given in the papers [7], [8]. Let $\lambda_\Lambda = \sum_{i=1}^k \lambda_i$ be the trace measure; from (4) we have

$$|l_{ij}(p)| \le l_{ii}(p) + l_{jj}(p) \le \sum_{s=1}^k l_{ss}(p) = \int_{D_l^n} p(z, \overline{z}) \mathrm{d}\lambda_\Lambda(z),$$

for all $p \in \wp$ with $p(z, \overline{z}) \ge 0$, when $z \in D_1^n$. Let $f \in \wp$ be an arbitrary polynomial. We then have $f(z, \overline{z}) = f_1(z, \overline{z}) + \mathrm{i} f_2(z, \overline{z})$, with $f_i(z, \overline{z})$ polynomials in $z, \overline{z}$ with real coefficients. If we consider the real coordinates x_j, y_j, when $z_j = x_j + \mathrm{i} y_j$, $1 \le j \le n$, we obtain

$$f_j(z, \overline{z}) = f_j^1(x_1, \ldots, x_n, y_1, \ldots, y_n) + \mathrm{i} f_j^2(x_1, \ldots, x_n, y_1, \ldots, y_n), \ j \in \{1, 2\},$$

with $f_j^k \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, for all $k, j \in \{1, 2\}$. As in the papers [3], [14], [19], there exist, for any $f_j^k \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, decompositions of the form

$$f_j^k(x_1, ..., x_n, y_1, ... y_n) = q_{j,1}^k(z, \overline{z}) - q_{j,2}^k(z, \overline{z}),$$

where

$$q^k_{j,r}(z,\overline{z}) = \sum_{j\in H\subset\mathbb{N}} [\alpha_j |p_j(z)|^2 \prod_{i=1}^{n}(1-|z_i|^2)^{k_i^j}], \tag{5}$$

with $p_j \in \wp_{an},\ k_i^j \in \mathbb{N},\ \alpha_j \in \mathbb{R},\ \alpha_j \geq 0,\ H$ finite, $\forall\, k,j,r \in \{1,2\}$. Hence we obtain

$$f_1(z,\overline{z}) = (q^1_{11} - q^1_{12}) + \mathrm{i}(q^2_{11} - q^2_{12})$$

and

$$f_2(z,\overline{z}) = (q^1_{21} - q^1_{22}) + \mathrm{i}(q^2_{21} - q^2_{22}),$$

with q^k_{jr}, for all $j,k,r \in \{1,2\}$, as in (5). Using these decompositions, we get

$$\begin{aligned} |l_{ij}(f_1)| &\leq |l_{ij}[(q^1_{11} - q^1_{12}) + \mathrm{i}(q^2_{11} - q^2_{12})]| \\ &\leq |l_{ij}(q^1_{11} - q^1_{12})| + |l_{ij}(q^2_{11} - q^2_{12})| \leq 2\int_{D_1^n} |f_1| \mathrm{d}\lambda_\Lambda(z). \end{aligned} \tag{6}$$

Similarly,

$$|l_{ij}(f_2)| \leq 2\int_{D_1^n} |f_2| \mathrm{d}\lambda_\Lambda(z). \tag{7}$$

From the inequalities (6) and (7) we obtain

$$|l_{ij}(f)| \leq |l_{ij}(f_1)| + |l_{ij}(f_2)| \leq 4\int_{D_1^n} |f| \mathrm{d}\lambda_\Lambda(z),\ \text{for any } f \in \wp. \tag{8}$$

Using the Hahn-Banach extension theorem in the complex case, we define l_{ij} on $L^1(\mathrm{d}\lambda_\Lambda)$, preserving the same inequality (8). The $\mathbb{C}$-linear functionals l_{ij}, $1 \leq i,j \leq k$, are bounded on $L^1(\mathrm{d}\lambda_\Lambda)$. Then there exist $g_{ij} \in L^\infty(\mathrm{d}\lambda_\Lambda)$ such that $l_{ij}(f) = \int_{D_1^n} f(z) g_{ij}(z) \mathrm{d}\lambda_\Lambda(z)$, for all $i,j \in \{1,\dots,k\}$. Because of this equality and assumption (i), we have

$$\begin{aligned} 0 \leq \sum_{1\leq i,j\leq k} l_{ij}(f) t_i \overline{t_j} &= \sum_{1\leq i,j\leq k} \int_{D_1^n} f(z) g_{ij}(z) t_i \overline{t_j} \mathrm{d}\lambda_\Lambda(z) \\ &= \int_{D_1^n} f(z) \sum_{1\leq i,j\leq k} g_{ij}(z) t_i \overline{t_j} d\lambda_\Lambda(z) \end{aligned}$$

for any $f \in \wp$, with $f(z,\overline{z}) \geq 0$ on D_1^n, and any $t_i, t_j \in \mathbb{C}$. With a routine measure theoretical argument, we get that the matrix $(g_{ij})_{1\leq i,j\leq k}$ is nonnegative $\mathrm{d}\lambda_\Lambda$ a.e. on D_1^n. If we take $\lambda_{ij} = g_{ij}\mathrm{d}\lambda_\Lambda$, for all $1 \leq i,j \leq k$, we obtain that the matrix $\Lambda = (\lambda_{ij})_{1\leq i,j\leq k}$ is positive defined on D_1^n, as required.

(ii)$\Rightarrow$(i). We assume the existance of a positive defined matrix of measures $\Lambda = (\lambda_{ij})_{1\leq i,j\leq k}$ on D_1^n for which we have the representations

$$\Gamma_{\alpha,\beta} = (s_{ij}(\alpha,\beta))_{1\leq i,j\leq k} = \int_{D_1^n} z^\alpha \overline{z}^\beta \mathrm{d}\Lambda(z),$$

for all $\alpha, \beta \in \mathbb{N}^n$. In these conditions, let g_{ij} be the Radon-Nikodym derivative of λ_{ij} with respect to the trace measure $\lambda_\Lambda = \sum_{i=1}^n \lambda_{ii}$. For the $\mathbb{C}$-linear map $L(.) = (l_{ij}(.))_{i,j=1}^k$ associated with $\Gamma_{\alpha,\beta}$ we then have

$$\sum_{i,j=1}^k l_{ij}(p) t_i \overline{t_j} = \int_{D_1^n} p(z,\overline{z}) (\sum_{i,j=1}^k g_{ij}(z) t_i \overline{t_j}) \mathrm{d}\lambda_\Lambda(z) \geq 0,$$

for any $p \in \wp$ with $p(z,\overline{z}) \geq 0$, when $z \in D_1^n$. Thus L is positive defined on D_1^n, as required in (i). □

REMARK 3.2. Let

$$\{\Gamma_{\alpha,\beta} = (s_{ij}(\alpha,\beta)_{1\leq i,j\leq k} \in M(k,\mathbb{C})\}_{\alpha,\beta \in \mathbb{N}^n, k\in\mathbb{N}^*}$$

be a multisequence of $k-$dimensional matrices with complex entries, for which the $\mathbb{C}$-linear map associated with it, denoted by $L : \wp \to M(k,\mathbb{C})$,

$$L(p) = [l_{ij}(p)]_{1\leq i,j\leq k} = \sum_{\alpha,\beta\in H} a_{\alpha\beta} [s_{ij}(\alpha,\beta)]_{1\leq i,j\leq k} = \sum_{\alpha,\beta\in H} a_{\alpha\beta} \Gamma_{\alpha,\beta},$$

when $p(z,\overline{z}) = \sum_{\alpha,\beta\in H} a_{\alpha\beta} z^\alpha \overline{z}^\beta$, H finite, $H \subset \mathbb{N}^n$, is positive definite on D_1^n (i.e., satisfies condition (i) of Proposition 3.1). Then ${}^t\overline{\Gamma}_{\alpha,\beta} = \Gamma_{\beta,\alpha}$, that is $s_{ij}(\alpha,\beta) = \overline{s_{ji}(\beta,\alpha)}$, for all $1 \leq i,j \leq k$ and all $\alpha,\beta \in \mathbb{N}^n$.

Proof. Since the $\mathbb{C}$-linear map $L = (l_{ij}(.))_{1\leq i,j\leq k}$ associated with $\{\Gamma_{\alpha,\beta}\}$ is positive on D_1^n, from assertion (i) of Proposition 3.1 we have $l_{ii}(p) \geq 0$, for all $1 \leq i \leq k$ and all $p \in \wp$ with $p(z,\overline{z}) \geq 0$, when $z \in D_1^n$. As in [3], [14], [19], and as in the proof of Proposition 3.1, all polynomials $p \in \wp$ admit a decomposition of the form $p(z,\overline{z}) = (q_1 - q_2) + \mathrm{i}(q_3 - q_4)$ with

$$q_s(z,\overline{z}) = \sum_{j\in H\subset\mathbb{N}} [\alpha_j |p_j(z)|^2 \prod_{i=1}^n (1 - |z_i|^2)^{k_i^j}],$$

with $p_j \in \wp_{an}$, $k_i^j \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$, $\alpha_j \geq 0$, H finite, $\forall\ s \in \{1,4\}$.

Let $p(z,\overline{z}) = z^\alpha \overline{z}^\beta = (q_1 - q_2) + \mathrm{i}(q_3 - q_4)$ be the corresponding decomposition.

Case a: $i = j$. Then we have

$$l_{ii}(z^\alpha \overline{z}^\beta) = [l_{ii}(q_1) - l_{ii}(q_2)] + \mathrm{i}[l_{ii}(q_3) - l_{ii}(q_4)]$$

and

$$\begin{aligned} l_{ii}(z^\beta \overline{z}^\alpha) &= [l_{ii}(\overline{q_1}) - l_{ii}(\overline{q_2})] - \mathrm{i}[l_{ii}(\overline{q_3}) - l_{ii}(\overline{q_4})] \\ &= [l_{ii}(q_1) - l_{ii}(q_2)] - \mathrm{i}[l_{ii}(q_3) - l_{ii}(q_4)]. \end{aligned}$$

From this we obtain

$$l_{ii}(z^\alpha \overline{z}^\beta) = \overline{l_{ii}(z^\beta \overline{z}^\alpha)} \Leftrightarrow s_{ii}(\alpha,\beta) = \overline{s_{ii}(\beta,\alpha)}.$$

Case b: $i \neq j$. Then, according to the equivalence of the assertions (i) and (ii) of Proposition 3.1, the following equalities hold

$$(\mathrm{Im}(l_{ij}(p)) = -\mathrm{Im}(l_{ji}(p)) \tag{1b}$$

and

$$(\mathrm{Re}(l_{ij}(p)) = \mathrm{Re}(l_{ji}(p))), \tag{2b}$$

when $p \in \wp$, with $p(z, \overline{z}) \geq 0$ on D_1^n. From the properties of the calculus with complex numbers, we have

$$\begin{aligned} \mathrm{Re}(l_{ij}(z^\alpha \overline{z}^\beta)) &= \mathrm{Re}[l_{ij}(q_1 - q_2) + \mathrm{i} l_{ij}(q_3 - q_4)] \\ &= \mathrm{Re}(l_{ij}(q_1)) - \mathrm{Re}(l_{ij}(q_2)) - \mathrm{Im}(l_{ij}(q_3)) + \mathrm{Im}(l_{ij}(q_4)) \end{aligned} \tag{9}$$

and

$$\mathrm{Im}(l_{ij}(z^\alpha \overline{z}^\beta)) = \mathrm{Im}(l_{ij}(q_1)) - \mathrm{Im}(l_{ij}(q_2) + \mathrm{Re}(l_{ij}(q_3)) - \mathrm{Re}(l_{ij}(q_4)). \tag{10}$$

Similarly,

$$\begin{aligned} \mathrm{Re}(l_{ji}(\overline{z}^\alpha z^\beta)) &= \mathrm{Re}[l_{ji}(q_1) - l_{ji}(q_2)] + Re[-\mathrm{i}(l_{ji}(q_3) - l_{ji}(q_4))] \\ &= \mathrm{Re}(l_{ji}(q_1)) - \mathrm{Re}(l_{ji}(q_2)) + \mathrm{Im}(l_{ji}(q_3)) - \mathrm{Im}(l_{ji}(q_4)) \end{aligned} \tag{9'}$$

and

$$\mathrm{Im}(l_{ji}(\overline{z}^\alpha z^\beta)) = \mathrm{Im}[l_{ji}(q_1)] - \mathrm{Im}[l_{ji}(q_2)] - \mathrm{Re}[l_{ji}(q_3)] + \mathrm{Re}[l_{ji}(q_4)]. \tag{10'}$$

Applying the equalities (1b) and (2b) in the relations (9) and (9′), respectively in (10) and (10′), we obtain

$$\mathrm{Re}(l_{ij}(z^\alpha \overline{z}^\beta)) = \mathrm{Re}(l_{ji}(\overline{z}^\alpha z^\beta))$$

and

$$\mathrm{Im}(l_{ij}(z^\alpha \overline{z}^\beta)) = -\mathrm{Im}((\overline{z}^\alpha z^\beta)),$$

that is

$$s_{ij}(\alpha, \beta) = \overline{s_{ji}(\beta, \alpha)},\ \forall 1 \leq i, j \leq k \Leftrightarrow \Gamma_{\alpha,\beta} =^t \overline{\Gamma}_{\beta,\alpha},\ \forall \alpha, \beta \in \mathbb{N}^n,$$

which is the required statement. □

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