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HOPF-GALOIS EXTENSIONS AND ISOMORPHISMS OF SMALL CATEGORIES

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Abstract. We associate two linear categories with two objects to a module over the subalgebra of coinvariants of a Hopf-Galois extension, and prove that they are isomorphic. The structure Theorem for cleft extensions, and the Militaru-Ştefan lifting Theorem can be obtained using these isomorphisms.

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INTRODUCTION

Our starting points are the following two classical results on Hopf algebras. The first one is the structure theorem of cleft H-comodule algebras [6], stating that a cleft H-comodule algebra is isomorphic to a crossed product, and, conversely, every crossed product is cleft. A comprehensive treatment can be found in [9, Ch. 7].

The second result is the Militaru-Ştefan lifting Theorem. Let A be a faithfully flat Hopf-Galois extension over its ring of coinvariants B, and M a B-module. Generalizing results due to Dade [4] on strongly graded rings, Militaru and Ştefan showed that the B-action on M can be extended to an A-action if and only if there exists an H-colinear algebra map between H and the A-endomorphism ring of $M \otimes_B A$.

Let us now explain the philosophy behind this note. A k-algebra can be viewed as a k-linear category with one object. Isomorphisms between k-algebras can be obtained from equivalences between k-linear categories. Examples of such equivalences come from faithfully flat Hopf algebra extensions: then we have a pair of inverse equivalences between modules over the ring of coinvariants and relative Hopf modules.

Now we consider "double" k-algebras, namely k-linear categories with two objects. For a right H-comodule algebra A, we introduce such a double algebra $\mathcal{C}_A$. One of its endomorphism algebras consists of k-linear maps from H to the coinvariants, and on its homomorphism modules consists of H-colinear maps $H \to A$. This construction is given in Section 2.

Given a module M over the coinvariants B, we introduce another double algebra $\mathcal{D}_M$, as the full subcategory of the category of B-modules and H-comodules, with objects $M \otimes H$ and $M \otimes_B A$. Our main result, Theorem 3.1 states that the categories $\mathcal{C}_A$ and $\mathcal{D}_M$ are isomorphic if A is a faithfully flat H-Galois extension of B. In Section 5, we discuss how this category equivalence (or at least some variation of it) can be applied the structure Theorem for cleft algebras, and in Section 6, we see how the Militaru-Ştefan lifting result can be obtained.

1. HOPF-GALOIS EXTENSIONS

Hopf-Galois theory was introduced in [3], and later generalized in [7, 10, 11]. We recall the definitions and the most important results. Let H be a Hopf algebra over a commutative ring k, and assume that the antipode S is bijective. We use the Sweedler notation for the comultiplication: $\Delta(h) = h_{(1)} \otimes h_{(2)}$, for $h \in H$. If M is a right H-comodule, then we use the following notation for the coaction ρ: $\rho(m) = m_{[0]} \otimes m_{[1]}$, for $m \in M$. In a similar way, we write $\lambda(n) = n_{[-1]} \otimes n_{[0]}$ for the left H-coaction on an element n in a left H-comodule N.

Let A be a right H-comodule algebra, this is an algebra in the monoidal category of right H-comodules. A relative right (A, H)-comodule is a right A-module that has also the structure of a right H-comodule such that the compatibility relation $\rho(ma) = m_{[0]}a_{[0]} \otimes m_{[1]}a_{[1]}$ holds for all $m \in M$ and $a \in A$. $M^{\mathrm{co}H} = \{m \in M \mid \rho(m) = m \otimes 1\}$ is the submodule of coinvariants, and is a right B-module, where $B = A^{\mathrm{co}A}$ is the subring of coinvariants of A. $\mathcal{M}_A^H$ is the category of relative Hopf modules, and right A-linear H-colinear maps. We have a pair of adjoint functors (F, G) between the categories $\mathcal{M}_B$ and $\mathcal{M}_A^H$. $F = -\otimes_B A$ is the induction functor, and $G = (-)^{\mathrm{co}A}$ is the coinvariants functor. The unit η and counit ε of the adjunction are the following ($M \in \mathcal{M}_B$ and $N \in \mathcal{M}_A^H$): $\eta_M : M \to (M \otimes_B A)^{\mathrm{co}A}$, $\eta_M(m) = m \otimes_B 1$; $\varepsilon_N : M^{\mathrm{co}A} \otimes A \to M$, $\varepsilon(m \otimes_B a) = ma$. The canonical map can associated to A is defined by

$$\mathrm{can} : A \otimes_B A \to A \otimes H, \ \mathrm{can}(a \otimes_B a') = aa'_{[0]} \otimes a'_{[1]}.$$

If can is an isomorphism, then A is called a *Hopf-Galois extension* or *H-Galois extension* of B.

We can also consider left-right (A, H)-modules: these are k-modules with a left A-action and a right H-coaction such that $\rho(am) = a_{[0]}m_{[0]} \otimes a_{[1]}m_{[1]}$, for all $a \in A$ and $m \in M$. We have a pair of adjoint functors $(F' = A \otimes_B -, G' = (-)^{\mathrm{co}H}$ between ${}_B\mathcal{M}$ and ${}_A\mathcal{M}^H$, the category of left-right (A, H)-modules. The unit and counit are this time given by $\eta'_M : M \to (A \otimes_B M)^{\mathrm{co}H}$, $\eta'_M(m) = 1 \otimes_B m$; $\varepsilon'_N : A \otimes_B N^{\mathrm{co}H} \to N$, $\varepsilon'_N(a \otimes_B n) = an$. The canonical map $\mathrm{can}' : A \otimes_B A \to A \otimes H$ is defined by the formula

$$\mathrm{can}'(a \otimes_B a') = a_{[0]}a' \otimes a_{[1]}.$$

It is well-known that can is an isomorphism if and only if can' is an isomorphism: this follows from the fact that $\mathrm{can}' = \Phi \circ \mathrm{can}$, with $\Phi : A \otimes H \to A \otimes H$ given by $\Phi(a \otimes h) = a_{[0]} \otimes a_{[1]}S(h)$ and $\Phi^{-1}(a \otimes h) = a_{[0]} \otimes a_{[1]}\overline{S}(h)$.

THEOREM 1.1. *Let A be a right H-comodule algebra, and consider the following statements:*

(1) (F,G) *is a pair of inverse equivalences;*
(2) (F,G) *is a pair of inverse equivalences and* $A \in {}_B\mathcal{M}$ *is flat;*
(3) can *is an isomorphism and* $A \in {}_B\mathcal{M}$ *is faithfully flat;*
(4) (F',G') *is a pair of inverse equivalences;*
(5) (F',G') *is a pair of inverse equivalences and* $A \in \mathcal{M}_B$ *is flat;*
(6) can' *is an isomorphism and* $A \in \mathcal{M}_B$ *is faithfully flat;*

Then $(3) \Longleftrightarrow (2) \Longrightarrow (1)$ *and* $(6) \Longleftrightarrow (5) \Longrightarrow (4)$. *If* H *is flat as a* k*-module, then* $(1) \Longleftrightarrow (2)$ *and* $(4) \Longleftrightarrow (5)$. *If* k *is a field, then the six statements are equivalent.*

Let A be a faithfully flat right H-Galois extension. The inverse of the canonical map can is completely determined by the map $\gamma_A = \mathrm{can}^{-1} \circ (\eta_A \otimes H) : H \to A \otimes_B A$, $h \mapsto \sum_i l_i(h) \otimes_B r_i(h)$. Then the element $\gamma_A(h)$ is characterized by the property

$$\sum_i l_i(h) r_i(h)_{[0]} \otimes r_i(h)_{[1]} = 1 \otimes h. \tag{1}$$

For all $h, h' \in H$ and $a \in A$, we have (see [11, 3.4]):

$$\gamma_A(h) \in (A \otimes_B A)^B; \tag{2}$$

$$\gamma_A(h_{(1)}) \otimes h_{(2)} = \sum_i l_i(h) \otimes_B r_i(h)_{[0]} \otimes r_i(h)_{[1]}; \tag{3}$$

$$\gamma_A(h_{(2)}) \otimes S(h_{(1)}) = \sum_i l_i(h)_{[0]} \otimes_B r_i(h) \otimes l_i(h)_{[1]}; \tag{4}$$

$$\sum_i l_i(h) r_i(h) = \varepsilon(h) 1_A; \tag{5}$$

$$\sum_i a_{[0]} l_i(a_{[1]}) \otimes_B r_i(a_{[1]}) = 1 \otimes_B a; \tag{6}$$

$$\sum_i l_i(\overline{S}(a_{[1]})) \otimes_B r_i(\overline{S}(a_{[1]})) a_{[0]} = a \otimes_B 1; \tag{7}$$

$$\gamma_A(hh') = \sum_{i,j} l_i(h') l_j(h) \otimes_B r_j(h) r_i(h'). \tag{8}$$

2. THE CATEGORIES $\mathcal{C}_A$ AND $\mathcal{C}'_A$

Let A be a right H-comodule algebra, and $B = A^{\mathrm{co}A}$, as in Section 1. We introduce a category $\mathcal{C}_A$, with two objects $\mathbf{1}$ and $\mathbf{2}$. The morphisms are defined as follows.

$$\begin{aligned} \mathcal{C}_A(\mathbf{1},\mathbf{1}) &= \mathrm{Hom}(H,B) \\ &= \{v : H \to A \mid \rho(v(h)) = v(h) \otimes 1, \text{ for all } h \in H\}; \\ \mathcal{C}_A(\mathbf{2},\mathbf{1}) &= \mathrm{Hom}^H(H,A) \\ &= \{t : H \to A \mid \rho(t(h)) = t(h_{(1)}) \otimes h_{(2)}, \text{ for all } h \in H\}; \end{aligned}$$

$$\begin{aligned}
\mathcal{C}_A(\mathbf{1},\mathbf{2}) &= \{u:\ H\to A \mid \rho(u(h)) = u(h_{(2)})\otimes S(h_{(1)}),\ \text{for all } h\in H\};\\
\mathcal{C}_A(\mathbf{2},\mathbf{2}) &= \{w:\ H\to A \mid \rho(w(h)) = w(h_{(2)})\otimes S(h_{(1)})h_{(3)},\ \text{for all } h\in H\}.
\end{aligned}$$

The composition of morphisms is given by the convolution on $\mathrm{Hom}(H,A)$. We have to verify that, for $f:\ \mathbf{i}\to\mathbf{j}$ and $g:\ \mathbf{j}\to\mathbf{k}$, $g*i\in\mathcal{C}_A(\mathbf{i},\mathbf{k})$. Let us do this in the case where $\mathbf{i}=\mathbf{j}=\mathbf{k}=\mathbf{2}$: for $w,w_1\in\mathcal{C}_A(\mathbf{2},\mathbf{2})$ and $h\in H$, we have

$$\begin{aligned}
\rho(w*w_1)(h) &= \rho(w(h_{(1)})w_1(h_{(2)})\\
&= w(h_{(2)})w_1(h_{(5)})\otimes S(h_{(1)})h_{(3)}S(h_{(4)})h_{(6)}\\
&= w(h_{(2)})w_1(h_{(3)})\otimes S(h_{(1)})h_{(4)}\\
&= (w*w_1)(h_{(2)})\otimes S(h_{(1)})h_{(3)},
\end{aligned}$$

and it follows that $w*w_1\in\mathcal{C}_A(\mathbf{2},\mathbf{2})$, as needed. Verification in all the other cases is similar and is left to the reader.

We also introduce the category $\mathcal{C}'_A$ and show that it is isomorphic to $\mathcal{C}_A$. It is introduced because it allows us to simplify slightly some of the computations in Section 3. $\mathcal{C}'_A$ also has two objects, $\mathbf{1}$ and $\mathbf{2}$. The morphisms are defined in the following fashion.

$$\begin{aligned}
\mathcal{C}'_A(\mathbf{1},\mathbf{1}) &= \mathrm{Hom}(H,B)\\
&= \{v': H\to A \mid \rho(v'(h)) = v'(h)\otimes 1,\ \text{for all } h\in H\};\\
\mathcal{C}'_A(\mathbf{1},\mathbf{2}) &= \mathrm{Hom}^H(H,A)\\
&= \{t': H\to A \mid \rho(t'(h)) = t'(h_{(1)})\otimes h_{(2)},\ \text{for all } h\in H\};\\
\mathcal{C}'_A(\mathbf{2},\mathbf{1}) &= \{u': H\to A \mid \rho(u'(h)) = u'(h_{(2)})\otimes \overline{S}(h_{(1)}),\ \text{for all } h\in H\};\\
\mathcal{C}'_A(\mathbf{2},\mathbf{2}) &= \{w': H\to A \,|\, \rho(w'(h)) = w'(h_{(2)})\otimes h_{(3)}\overline{S}(h_{(1)}),\ \text{for all } h\in H\}.
\end{aligned}$$

The composition of two morphisms in $\mathcal{C}'_A$ is given by the convolution product in $\mathrm{Hom}(H^{\mathrm{cop}},A)$: $(f'\star g')(h) = f'(h_{(2)})g'(h_{(1)})$.

PROPOSITION 2.1. *We have an isomorphism of categories $\gamma:\ \mathcal{C}'_A\to\mathcal{C}_A$, which is the identity at the level of objects. At the level of morphisms, it is given by $\gamma(f') = f'\circ S$.*

Proof. We have to show first that $\gamma(\mathcal{C}'_A(\mathbf{i},\mathbf{j}))\subset\mathcal{C}_A(\mathbf{i},\mathbf{j})$. Let us do this in the case $\mathbf{i}=\mathbf{j}=\mathbf{2}$, the other cases are done in a similar way. So take $w'\in\mathcal{C}'_A(\mathbf{2},\mathbf{2})$, and let $w = w'\circ S = \gamma(w')$. Then for all $h\in H$, we have that

$$\begin{aligned}
\rho(w(h)) &= \rho(w'(S(h))) = w'(S(h)_{(2)})\otimes S(h)_{(3)}\overline{S}(S(h)_{(1)})\\
&= w'(S(h_{(2)}))\otimes S(h_{(1)})h_{(3)} = w(h_{(2)})\otimes S(h_{(1)})h_{(3)},
\end{aligned}$$

proving that $w\in\mathcal{C}_A(\mathbf{2},\mathbf{2})$, as needed. It is easy to see that γ respects the composition of morphisms:

$$\begin{aligned}
\gamma(f'\star g')(h) &= (f'\star g')(S(h)) = f'(S(h_{(1)}))g'(S(h_{(2)}))\\
&= f(h_{(1)})g(h_{(2)}) = (f*g)(h).
\end{aligned}$$

Finally, γ is an isomorphism. The inverse functor $\overline{\gamma}$ is given by $\overline{\gamma}(f) = f\circ\overline{S}$. $\square$

The functor γ induces maps $\gamma_{ji}:\ \mathcal{C}'_A(\mathbf{i},\mathbf{j}) \to \mathcal{C}_A(\mathbf{i},\mathbf{j})$.

3. THE MAIN RESULT

Let A be a faithfully flat right H-Galois extension. We assume moreover that H is projective as a k-module. This is always satisfied if we work over a field k. Let P and Q be two right relative Hopf modules. We have a map

$$\rho:\ \mathrm{Hom}_A(P,Q) \to \mathrm{Hom}_A(P,Q\otimes H),\ \ \rho(f)(p) = f(p_{[0]})_{[0]} \otimes f(p_{[0]})_{[1]} S(p_{[1]}).$$

As H is projective, the natural map $\mathrm{Hom}_A(P,Q)\otimes H \to \mathrm{Hom}_A(P,Q\otimes H)$ is a monomorphism, and we can consider $\mathrm{Hom}_A(P,Q)\otimes H$ as a submodule of $H \to \mathrm{Hom}_A(P,Q\otimes H)$. We call $f \in \mathrm{Hom}_A(P,Q)$ *rational* if $\rho(f) \in \mathrm{Hom}_A(P,Q)\otimes H$, that is, if there exists an element $f_{[0]} \otimes f_{[1]} \in \mathrm{Hom}_A(P,Q) \otimes H$ (summation implicitely understood) such that $\rho(f)(p) = f_{[0]}(p) \otimes f_{[1]}$, for all $p \in P$, which is equivalent to

$$\rho(f(p)) = f_{[0]}(p_{[0]}) \otimes f_{[1]} p_{[1]}. \tag{9}$$

The submodule of $\mathrm{Hom}_A(P,Q)$ consisting of all rational maps is denoted by $\mathrm{HOM}_A(P,Q)$, and is a right H-comodule. $\mathrm{END}_A(P)$ is a right H-comodule algebra. Now we take $P = M \otimes_B A$, where $M \in \mathcal{M}_B$, $E = \mathrm{END}_A(M \otimes_B A)$ and $F = E^{\mathrm{co}H} = \mathrm{END}_A^H(M\otimes_B A) \cong \mathrm{End}_B(M)$, in view of Theorem 1.1. Then we can consider the categories $\mathcal{C}_E$ and $\mathcal{C}'_E$, as in Section 2.

We have seen in Section 1 that $M \otimes_B A \in \mathcal{M}_A^H$ is a relative Hopf module. In particular, it is also an object in $\mathcal{M}_B^H$, where B is considered as a right H-comodule algebra with trivial H-coaction. In fact $\mathcal{M}_B^H$ is the category of right B-modules with a right H-coaction such that $\rho(mb) = m_{[0]} \otimes m_{[1]}b$, for all $m \in M$ and $b \in B$. $M \otimes H$ is also an object of $\mathcal{M}_B^H$, with B-action and H-coaction given by $\rho(m\otimes h) = m \otimes \Delta(h)$ and $(m\otimes h)b = mb \otimes h$.

Now let $\mathcal{D}_M$ be the full subcategory of $\mathcal{M}_B^H$ with objects $M \otimes_B A$ and $M \otimes H$. Out main result is the following.

THEOREM 3.1. *Let H be a projective Hopf algebra, and A a faithfully flat right H-Galois extension. For $M \in \mathcal{M}_B$, we have a commutative diagram of isomorphisms of categories:*

$$\begin{tikzcd}
\mathcal{C}'_E \arrow[rr, "\gamma"] \arrow[dr, "\beta"'] & & \mathcal{C}_E \arrow[dl, "\alpha"] \\
& \mathcal{D}_M &
\end{tikzcd}$$

At the level of morphisms, the functors α and α' are defined in the obvious way: $\alpha(\mathbf{1}) = \alpha'(\mathbf{1}) = M \otimes H$; $\alpha(\mathbf{2}) = \alpha'(\mathbf{2}) = M \otimes_B A$. In the subsequent Lemmas, we will define α and α' at the level of morphisms. The proof of the following result is straightforward, and is left to the reader.

Lemma 3.2. *We have an isomorphism of k-modules*

$$\delta_1:\ \mathrm{Hom}_B(M\otimes_B A, M)\to \mathrm{Hom}_B^H(M\otimes_B A, M\otimes H),$$

given by $\delta_1(\phi)(m\otimes_B a)=\phi(m\otimes_B a_{[0]})\otimes a_{[1]}$*;* $\overline{\delta}_1(\varphi)=(M\otimes\varepsilon)\circ\varphi$*. We have an isomorphism of k-algebras*

$$\delta_2:\ \mathrm{Hom}_B(M\otimes H, M)\to \mathrm{End}_B^H(M\otimes H),$$

given by $\delta_2(\Theta)(m\otimes h)=\Theta(m\otimes h_{(1)})\otimes h_{(2)}$*;* $\overline{\delta}_2(\theta)=(M\otimes\varepsilon)\circ\theta$*. The multiplication on* $\mathrm{Hom}_B(M\otimes H,M)$ *is given by the formula* $\Theta\cdot\Theta'=\Theta\circ\delta_2(\Theta')$*, or, more explicitly,*

$$(\Theta\cdot\Theta')(m\otimes h)=\Theta(\Theta'(m\otimes h_{(1)})\otimes h_{(2)}). \tag{10}$$

Lemma 3.3. *We have an algebra map*

$$\tilde\beta_{11}:\ \mathcal{C}'_E(\mathbf{1},\mathbf{1})=\mathrm{Hom}(H,F)\to \mathrm{Hom}_B(M\otimes H,M),$$

given by $\tilde\beta_{11}(v')(m\otimes h)=\eta_M^{-1}\big(v'(h)(m\otimes_B 1)\big)$*.*

Proof. For all $h\in H$, we have that $v'(h)\in F=E^{\mathrm{co}H}$. Using (9), we find that $\rho\big(v'(h)(m\otimes_B 1)\big)=v'(h)(m\otimes_B 1)\otimes 1$, hence $v'(h)(m\otimes_B 1)\in (M\otimes_B A)^{\mathrm{co}H}$. We know from Theorem 1.1 that $\eta_M:\ M\to (M\otimes_B A)^{\mathrm{co}H}$ is an isomorphism, so that $\tilde\beta_{11}$ is well-defined, and is characterized by the formula

$$\tilde\beta_{11}(v')(m\otimes h)\otimes_B 1=v'(h)(m\otimes_B 1). \tag{11}$$

Let us now show that $\tilde\beta_{11}(v')$ is right B-linear. For all $m\in M$, $b\in B$ and $h\in H$, we have

$$\begin{aligned}\tilde\beta_{11}(v')(mb\otimes h)\otimes_B 1&=v'(h)(mb\otimes_B 1)=v'(h)(m\otimes_B 1)b\\ &= \tilde\beta_{11}(v')(m\otimes h)\otimes_B b=\tilde\beta_{11}(v')(m\otimes h)b\otimes_B 1.\end{aligned}$$

We will now show that $\tilde\beta_{11}$ has an inverse, given by

$$\big(\hat\beta_{11}(\Theta)(h)\big)(m\otimes_B a)=\Theta(m\otimes h)\otimes_B a.$$

We have to show first that $\hat\beta_{11}$ is well-defined, that is, $\hat\beta_{11}(h)\in F$, for all $h\in H$. To this end, we compute that

$$\begin{aligned}\rho\Big(\big(\hat\beta_{11}(\Theta)(h)\big)(m\otimes_B a)\Big)&=\Theta(m\otimes h)\otimes_B a_{[0]}\otimes a_{[1]}\\ &= \big(\hat\beta_{11}(\Theta)(h)\big)(m\otimes_B a_{[0]})\otimes a_{[1]},\end{aligned}$$

and conclude from (9) that $\rho\big(\hat\beta_{11}(\Theta)(h)\big)=\hat\beta_{11}(\Theta)(h)\otimes 1$.

We now show that $\tilde\beta_{11}$ and $\hat\beta_{11}$ are inverses. For all $\Theta\in\mathrm{Hom}_B(M\otimes H,M)$, $v'\in\mathrm{Hom}(H,F)$ $m\in M$, $h\in H$ and $a\in A$, we have

$$\begin{aligned}\tilde\beta_{11}\big(\hat\beta_{11}(\Theta)\big)(m\otimes h)\otimes_B 1&=\big(\hat\beta_{11}(\Theta)(h)\big)(m\otimes_B 1)\\ &= \Theta(m\otimes h)\otimes_B 1;\\ \big(\hat\beta_{11}(\tilde\beta_{11}(v'))(h)\big)(m\otimes_B a)&=(\tilde\beta_{11}(v'))(m\otimes h)\otimes_B a\end{aligned}$$

$$= \quad v'(h)(m \otimes_B 1)a = v'(h)(m \otimes_B 1).$$

Let us finally show that $\tilde{\beta}_{11}$ is an algebra map. For $v', v'_1 : H \to F$, $m \in M$ and $h \in H$, we have

$$\begin{aligned}
\big(\tilde{\beta}_{11}(v') \cdot \tilde{\beta}_{11}(v'_1)\big)(m \otimes h) \otimes_B 1 &= \tilde{\beta}_{11}(v')\big(\tilde{\beta}_{11}(v'_1)(m \otimes h_{(1)}) \otimes h_{(2)}\big) \otimes 1 \\
&= v'(h_{(2)})\big(\tilde{\beta}_{11}(v'_1)(m \otimes h_{(1)}) \otimes_B 1\big) = (v'(h_{(2)}) \circ v'_1(h_{(1)}))(m \otimes_B 1) \\
&= (v' \star v'_1)(h)(m \otimes_B 1) = \tilde{\beta}_{11}(v' \star v'_1)(m \otimes h) \otimes_B 1,
\end{aligned}$$

and it follows that $\tilde{\beta}_{11}(v' \star v'_1) = \tilde{\beta}_{11}(v') \cdot \tilde{\beta}_{11}(v'_1)$. □

COROLLARY 3.4. *We have algebra isomorphisms*

$$\beta_{11} = \delta_2 \circ \tilde{\beta}_{11} : \ \mathcal{C}'_E(\mathbf{1}, \mathbf{1}) \to \mathrm{End}^H_B(M \otimes H);$$

$$\alpha_{11} = \delta_2 \circ \tilde{\beta}_{11} \circ \gamma_{11}^{-1} : \ \mathcal{C}_E(\mathbf{1}, \mathbf{1}) \to \mathrm{End}^H_B(M \otimes H).$$

LEMMA 3.5. *We have an isomorphism of k-modules*

$$\beta_{21} : \ \mathcal{C}'_E(\mathbf{1}, \mathbf{2}) = \mathrm{Hom}(H, E) \to \mathrm{Hom}^H_B(M \otimes H, M \otimes_B A),$$

given by $\beta_{21}(t')(m \otimes h) = t'(h)(m \otimes_B 1)$, for $t' \in \mathrm{Hom}(H, E)$, $m \in M$, $h \in H$. Consequently, we also have an isomorphism

$$\alpha_{21} = \beta_{21} \circ \gamma_{21}^{-1} : \ \mathcal{C}_E(\mathbf{1}, \mathbf{2}) \to \mathrm{Hom}^H_B(M \otimes H, M \otimes_B A).$$

Proof. It is easy to see that $\beta_{21}(t')$ is right A-linear:

$$\begin{aligned}
\beta_{21}(t')(mb \otimes h) &= t'(h)(mb \otimes_B 1) = t'(h)(m \otimes_B b) \\
&= t'(h)(m \otimes_B 1)b = (\beta_{21}(t')(m \otimes h))b.
\end{aligned}$$

$\beta_{21}(t')$ is right H-colinear:

$$\begin{aligned}
\rho\big(\beta_{21}(t')(m \otimes h)\big) &= \rho\big(t'(h)(m \otimes_B 1)\big) = t'(h)_{[0]}(m \otimes_B 1) \otimes t'(h)_{[1]} \\
&= t'(h_{(1)})(m \otimes_B 1) \otimes h_{(2)} = \beta_{21}(t')(m \otimes h_{(1)}) \otimes h_{(2)}.
\end{aligned}$$

This shows that $\beta_{21}(t') \in \mathrm{Hom}^H_B(M \otimes H, M \otimes_B A)$, as needed. Now we define a map

$$\overline{\beta}_{21} : \ \mathrm{Hom}^H_B(M \otimes H, M \otimes_B A) \to \mathrm{Hom}(H, E)$$

by the formula $(\overline{\beta}_{21}(\psi))(h)(m \otimes_B a) = \psi(m \otimes h)a$. We first show that $\overline{\beta}_{21}$ is well-defined, and then that it is inverse to β_{21}.

$\overline{\beta}_{21}(\psi)$ is right H-colinear: we first compute

$$\begin{aligned}
\rho\big((\overline{\beta}_{21}(\psi))(h)(m \otimes_B a)\big) &= \rho\big(\psi(m \otimes h)a\big) \\
&= \psi(m \otimes h_{(1)})a_{[0]} \otimes h_{(2)}a_{[1]} \\
&= (\overline{\beta}_{21}(\psi))(h_{(1)})(m \otimes_B a_{[0]}) \otimes h_{(2)}a_{[1]},
\end{aligned}$$

and we conclude from (9) that $\rho\big((\overline{\beta}_{21}(\psi))(h)\big) = (\overline{\beta}_{21}(\psi))(h_{(1)}) \otimes h_{(2)}$, as needed.

Let us finally show that β_{21} and $\overline{\beta}_{21}$ are inverses. For all $t' \in \mathrm{Hom}^H(H, E)$, $\psi \in \mathrm{Hom}^H_B(M \otimes H, M \otimes_B A)$, $m \in M$, $a \in A$ and $h \in H$, we have

$$\begin{aligned}(\beta_{21} \circ \overline{\beta}_{21})(\psi)(m \otimes h) &= (\overline{\beta}_{21}(h))(m \otimes_B 1) \\ &= \psi(m \otimes_B 1)a = \psi(m \otimes_B a); \\ \Big(((\overline{\beta}_{21} \circ \beta_{21})(t'))(h)\Big)(m \otimes_B a) &= (\beta_{21}(t'))(m \otimes h)a \\ &= t'(h)(m \otimes_B 1)a = t'(h)(m \otimes_B a).\end{aligned}$$

□

Lemma 3.6. *We have an isomorphism of k-modules*

$$\tilde{\beta}_{12}:\ \mathcal{C}'_E(\mathbf{2}, \mathbf{1}) \to \mathrm{Hom}_B(M \otimes_B A, M),$$

given by $\tilde{\beta}_{12}(u')(m \otimes_B a) = \eta_M^{-1}\big(u'(a_{[1]})(m \otimes_B a_{[0]})\big)$.

Proof. First, we have to show that $u'(a_{[1]})(m \otimes_B a_{[0]}) \in (M \otimes_B A)^{\mathrm{co}H}$. This can be seen as follows:

$$\begin{aligned}\rho\big(u'(a_{[1]})(m \otimes_B a_{[0]})\big) &= u'(a_{(3)})(m \otimes_B a_{[0]}) \otimes \overline{S}(a_{(2)})a_{[1]} \\ &= u'(a_{[1]})(m \otimes_B a_{[0]}) \otimes 1.\end{aligned}$$

Remark that $\tilde{\beta}_{12}(u')(m \otimes_B a)$ is characterized by the formula

$$\tilde{\beta}_{12}(u')(m \otimes_B a) \otimes_B 1 = u'(a_{[1]})(m \otimes_B a_{[0]}). \tag{12}$$

Now we show that $\tilde{\beta}_{12}(u')$ is right B-linear: for $b \in B$, we have

$$\begin{aligned}\tilde{\beta}_{12}(u')(m \otimes_B ab) \otimes_B 1 &= u'(a_{[1]})(m \otimes_B a_{[0]}b) = u'(a_{[1]})(m \otimes_B a_{[0]})b \\ &= \tilde{\beta}_{12}(u')(m \otimes_B a) \otimes_B b = \tilde{\beta}_{12}(u')(m \otimes_B a)b \otimes_B 1.\end{aligned}$$

Now we construct a map

$$\hat{\alpha}_{12} : \mathrm{Hom}_B(M \otimes_B A, M) \to \mathcal{C}_E(\mathbf{2}, \mathbf{1}) = \mathrm{Hom}^H(H, E).$$

as follows:

$$\big(\hat{\alpha}_{12}(\phi)(h)\big)(m \otimes_B a) = \sum_i \phi(m \otimes l_i(h)) \otimes_B r_i(h)a. \tag{13}$$

It is clear that $\big(\hat{\alpha}_{12}(\phi)\big)(h)$ is right A-linear. Then we need to show that $\hat{\alpha}_{12}(\phi)$ is right H-colinear. To this end, we need to show that

$$\rho\big(\hat{\alpha}_{12}(\phi)(h)\big) = \hat{\alpha}_{12}(\phi)(h_{(1)}) \otimes h_{(2)}, \tag{14}$$

for all $h \in H$. For all $m \in M$ and $a \in A$, we compute

$$\begin{aligned}&\rho\Big(\big(\hat{\alpha}_{12}(\phi)(h)\big)(m \otimes_B a)\Big) \\ &\quad= \sum_i \phi(m \otimes l_i(h)) \otimes_B r_i(h)_{[0]}a_{[0]} \otimes r_i(h)_{[1]}a_{[1]} \\ &\quad\overset{(3)}{=} \sum_i \phi(m \otimes l_i(h_{(1)})) \otimes_B r_i(h_{(1)})a_{[0]} \otimes h_{(2)}a_{[1]}\end{aligned}$$

$$= \big(\hat{\alpha}_{12}(\phi)(h_{(1)})\big)(m \otimes_B a_{[0]}) \otimes h_{(2)}a_{[1]},$$

and (14) follows as an application of (9).

Now we define $\hat{\beta}_{12} = \hat{\alpha}_{12} \circ \gamma_{12}^{-1}$, and show that $\hat{\beta}_{12}$ and $\hat{\alpha}_{12}$ are inverses. $\hat{\beta}_{12}$ is given by the formula

$$(\hat{\beta}_{12}(\phi)(h))(m \otimes_B a) = \sum_i \phi(m \otimes_B l_i(\overline{S}(h)) \otimes_B r_i(\overline{S}(h))a.$$

Now we compute

$$\begin{aligned}
&\Big(\big((\hat{\beta}_{12} \circ \tilde{\beta}_{12})(u')\big)(h)\Big)(m \otimes_B a)\\
&\quad= \sum_i (\tilde{\beta}_{12}(u'))\big(m \otimes_B l_i(\overline{S}(h)\big) \otimes_B r_i(\overline{S}(h))a\\
&\quad= (u'(l_i(\overline{S}(h)_{[1]}))\big(m \otimes_B l_i(\overline{S}(h)_{[0]}\big) r_i(\overline{S}(h)a\\
&\quad\overset{(4)}{=} (u'(S(\overline{S}(h_{(2)}))))\big(m \otimes_B l_i(\overline{S}(h_{(1)})\big) r_i(\overline{S}(h_{(1)})a\\
&\quad= (u'(h_{(2)}))\big(m \otimes_B l_i(\overline{S}(h_{(1)}) r_i(\overline{S}(h_{(1)})a\big)\\
&\quad\overset{(5)}{=} u'(h)(m \otimes_B a);
\end{aligned}$$

$$\begin{aligned}
&\big((\tilde{\beta}_{12} \circ \hat{\beta}_{12})(\phi)\big)(m \otimes_B a) \otimes_B 1\\
&\quad= (\hat{\beta}_{12}(\phi))(a_{[1]})(m \otimes_B a_{[0]})\\
&\quad= \sum_i \phi(m \otimes_B l_i(\overline{S}(a_{[1]}))) \otimes_B r_i(\overline{S}(a_{[1]}))a_{[0]}\\
&\quad\overset{(7)}{=} \phi(m \otimes_B a) \otimes_B 1.
\end{aligned}$$

□

COROLLARY 3.7. *We have the k-module isomorphisms*

$$\beta_{12} = \delta_1 \circ \tilde{\beta}_{12} : \ \mathcal{C}'_E(\mathbf{2},\mathbf{1}) \to \mathrm{Hom}_B^H(M \otimes_B A, M \otimes H));$$

$$\alpha_{12} = \delta_1 \circ \tilde{\beta}_{12} \circ \gamma_{12}^{-1} : \ \mathcal{C}_E(\mathbf{2},\mathbf{1}) \to \mathrm{Hom}_B^H(M \otimes_B A, M \otimes H)).$$

LEMMA 3.8. *We have an algebra isomorphism $\beta_{22} : \mathcal{C}'_E(\mathbf{2},\mathbf{2}) \to \mathrm{End}_B^H(M \otimes_B A)$, given by the formula $(\beta_{22}(w'))(p) = w'(p_{[1]})(p_{[0]})$, for all $p \in M \otimes_B A$. Consequently, we also have an algebra isomorphism $\alpha_{22} = \beta_{22} \circ \gamma_{22}^{-1} : \mathcal{C}'_E(\mathbf{2},\mathbf{1}) \to \mathrm{End}_B^H(M \otimes_B A)$.*

Proof. We first show that $\beta_{22}(w')$ is right B-linear. For $p \in M \otimes_B A$ and $b \in B$, we have $\rho(pb) = p_{[0]}b \otimes p_{[1]}$, and

$$(\beta_{22}(w'))(pb) = w'(p_{[1]})(p_{[0]}b) = w'(p_{[1]})(p_{[0]})b.$$

$\beta_{22}(w')$ is right H-co-linear. Since $w' \in \mathcal{C}'_E(\mathbf{2},\mathbf{1})$, we have

$$\rho(w'(h)) = w(h_{(2)}) \otimes h_{(3)}\overline{S}(h_{(1)}),$$

hence

$$\rho(w'(h)(p)) = w(h_{(2)})(p_{[0]}) \otimes h_{(3)}\overline{S}(h_{(1)})p_{[1]}. \tag{15}$$

Now we have

$$\begin{aligned}\rho\big((\beta_{22}(w'))(p)\big) &= \rho\big(w'(p_{[1]})(p_{[0]})\big) \overset{(15)}{=} w'(p_{(3)})(p_{[0]}) \otimes p_{[4]}\overline{S}(p_{[2]})p_{[1]}\\ &= w'(p_{(1)})(p_{[0]}) \otimes p_{[2]} = (\beta_{22}(w'))(p_{[0]}) \otimes p_{[1]}.\end{aligned}$$

We next show that β_{22} is an algebra morphism, that is, it preserves multiplication and unit. Mulitplication:

$$\begin{aligned}(\beta_{22}(w' \star w'_1))(p) &= ((w' \star w'_1)(p_{[1]})(p_{[0]})\\ &= (w'(p_{[2]}) \circ w'_{[1]}(p_{[1]}))(p_{[0]})\\ &= w'(p_{[1]})\big(\beta_{22}(w'_1)(p_{[0]})\big)\\ &= w'\Big(\big(\beta_{22}(w'_1)(p)\big)_{[1]}\Big)\Big(\big(\beta_{22}(w'_1)(p)\big)_{[0]}\Big)\\ &= \beta_{22}(w')\big(\beta_{22}(w'_1)(p)\big)\\ &= (\beta_{22}(w') \circ \beta_{22}(w'_1))(p).\end{aligned}$$

In the fourth equality we used the fact that $\beta_{22}(w'_1)$ is right H-colinear.

Unit: $(\beta_{22}(\eta_E \circ \varepsilon_H))(p) = (\eta(\varepsilon(p_{[1]}))(p_{[0]}) = p$.

Now we consider the map $\overline{\alpha}_{22} :\ \mathrm{End}_B^H(M \otimes_B A) \to \mathcal{C}_E(\mathbf{2},\mathbf{2})$, defined as follows: for $\kappa \in \mathrm{End}_B^H(M \otimes_B A)$, let

$$\big(\overline{\alpha}_{22}(\kappa)(h)\big)(m \otimes_B a) = \sum_i \kappa(m \otimes_B l_i(h))r_i(h)a.$$

We have to show that $\overline{\alpha}_{12}(\kappa) \in \mathcal{C}_E(\mathbf{2},\mathbf{2})$, that is,

$$\rho\big(\overline{\alpha}_{12}(\kappa)(h)\big) = \big(\overline{\alpha}_{12}(\kappa)(h_{(2)})\big) \otimes S(h_{(1)})h_{(3)}. \tag{16}$$

We proceed as follows: for all $m \in M$ and $a \in A$, we have

$$\begin{aligned}\rho\big(\overline{\alpha}_{12}(\kappa)(h)(m \otimes_B a\big) &= \rho\big(\sum_i \kappa(m \otimes_B l_i(h))r_i(h)a\big)\\ &= \sum_i \kappa\big(m \otimes_B l_i(h)_{[0]}\big)r_i(h)_{[0]}a_{[0]} \otimes l_i(h)_{[1]}r_i(h)_{[1]}a_{[1]}\\ &\overset{(3)}{=} \sum_i \kappa\big(m \otimes_B l_i(h_{(1)})_{[0]}\big)r_i(h_{(1)})a_{[0]} \otimes l_i(h_{(1)})_{[1]}h_{(2)}a_{[1]}\\ &\overset{(4)}{=} \sum_i \kappa\big(m \otimes_B l_i(h_{(2)})\big)r_i(h_{(2)})a_{[0]} \otimes S(h_{(1)})h_{(3)}a_{[1]}\\ &= \overline{\alpha}_{22}(\kappa)(h_{(2)})(m \otimes_B a_{[0]}) \otimes S(h_{(1)})h_{(3)}a_{[1]}\end{aligned}$$

In the second equality, we used that κ is right H-colinear. (16) then follows as an application of (9). Let us now show that $\overline{\beta}_{22} = \overline{\alpha}_{12} \circ \gamma_{22}^{-1}$ and β_{22} are inverses.

$$\big((\beta_{22} \circ \overline{\beta}_{22})(\kappa)\big)(m \otimes_B a) = (\overline{\beta}_{22}(\kappa)(a_{[1]}))(m \otimes_B a_{[0]})$$

$$
\begin{aligned}
&= \kappa\big(m\otimes_B l_i(\overline{S}(a_{[1]}))r_i(\overline{S}(a_{[1]})a_{[0]} \overset{(7)}{=} \kappa(m\otimes_B a);\\
&\big(((\overline{\beta}_{22}\circ\beta_{22})(w'))(h)\big)(m\otimes_B a)\\
&= \sum_i \beta_{22}(w')(m\otimes_B l_i(\overline{S}(h)))r_i(\overline{S}(h))a\\
&= \sum_i \big(w'\big(l_i(\overline{S}(h))_{[1]}\big)\big)\big(m\o_B l_i(\overline{S}(h))_{[0]}\big)r_i(\overline{S}(h))a\\
&\overset{(4)}{=} \sum_i w'\big(S(\overline{S}(h_{(2)})\big)\big(m\otimes_B l_i(\overline{S}(h_{(1)})\big)r_i(\overline{S}(h_{(1)})a\\
&= \sum_i w'(h_{(2)})\big(m\otimes_B l_i(\overline{S}(h_{(1)})r_i(\overline{S}(h_{(1)})a\big)\\
&\overset{(5)}{=} w'(h)(m\otimes_B a).
\end{aligned}
$$

□

Proof. (*of Theorem 3.1*). In the preceding Lemmas, we have shown that there exist isomorphisms

$$\mathcal{C}'_E(\mathbf{i},\mathbf{j}) \xrightarrow{\gamma_{ji}} \mathcal{C}_E(\mathbf{i},\mathbf{j}) \xrightarrow{\alpha_{ji}} \mathrm{Hom}_B^H(\alpha(\mathbf{i}),\alpha(\mathbf{j}))$$

The proof of Theorem 3.1 will be finished if we can show that, given $f:\ \mathbf{i}\to\mathbf{j}$ and $g:\ \mathbf{j}\to\mathbf{k}$ in $\mathcal{C}_E$, we have

$$\alpha_{kj}(g)\circ\alpha_{ji}(f) = \alpha_{ki}(g*f) \tag{17}$$

We already know that (17) holds if $\mathbf{i}=\mathbf{j}=\mathbf{k}$, see Corollary 3.4 and Lemma 3.8.

We now fix the following notation.

$$
\begin{array}{lll}
v'\in\mathcal{C}'_E(\mathbf{1},\mathbf{1}) & v=\gamma_{11}(v')\in\mathcal{C}_E(\mathbf{1},\mathbf{1}) & \theta=\alpha_{11}(v):\ M\otimes H\to M\otimes H\\
t'\in\mathcal{C}'_E(\mathbf{1},\mathbf{2}) & u=\gamma_{21}(t')\in\mathcal{C}_E(\mathbf{1},\mathbf{2}) & \psi=\alpha_{21}(v):\ M\otimes H\to M\otimes_B A\\
u'\in\mathcal{C}'_E(\mathbf{2},\mathbf{1}) & t=\gamma_{12}(u')\in\mathcal{C}_E(\mathbf{2},\mathbf{1}) & \varphi=\alpha_{12}(v):\ M\otimes_B A\to M\otimes H\\
w'\in\mathcal{C}'_E(\mathbf{2},\mathbf{2}) & w=\gamma_{22}(w')\in\mathcal{C}_E(\mathbf{2},\mathbf{2}) & \kappa=\alpha_{22}(w):\ M\otimes_B A\to M\otimes_B A
\end{array}
$$

Furthermore, let $\Theta=\overline{\delta}_2(\theta)$ and $\phi=\overline{\delta}_1(\varphi)$, see Lemma 3.2. The six remaining identities that we have to prove are

$$\alpha_{21}(v*u) = \alpha_{21}(u)\circ\alpha_{11}(v)=\psi\circ\theta; \tag{18}$$
$$\alpha_{21}(w*u) = \alpha_{22}(w)\circ\alpha_{21}(u)=\kappa\circ\psi; \tag{19}$$
$$\alpha_{11}(t*u) = \alpha_{12}(t)\circ\alpha_{21}(u)=\varphi\circ\psi; \tag{20}$$
$$\alpha_{12}(t*w) = \alpha_{12}(t)\circ\alpha_{22}(w)=\varphi\circ\kappa; \tag{21}$$
$$\alpha_{12}(v*t) = \alpha_{11}(v)\circ\alpha_{12}(t)=\theta\circ\varphi; \tag{22}$$
$$\alpha_{22}(u*t) = \alpha_{21}(u)\circ\alpha_{12}(t)=\psi\circ\varphi;. \tag{23}$$

(18) is equivalent to $\overline{\beta}_{21}(\psi\circ\theta)=t'\star v'$. This can be shown as follows

$$
\begin{aligned}
((t'\star v')(h))(m\otimes_B a) &= (t'(h_{(2)})\circ v'(h_{(1)}))(m\otimes_B a)\\
&= (t'(h_{(2)}))\big(\Theta(m\otimes h_{(1)})\otimes_B a\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \psi\big(\Theta(m\otimes h_{(1)})\otimes h_{(2)}\big)a\\
&= (\psi\circ\theta)(m\otimes h)a\\
&= \big(\overline{\beta}_{21}(\psi\circ\theta)\big)(m\otimes_B a).
\end{aligned}
$$

(19) is equivalent to $\beta_{21}(w'\star t')=\kappa\circ\psi$.

ψ is given by the formula (see Lemma 3.5): $\psi(m\otimes h)=t'(h)(m\otimes_B 1)$.

t' is right H-colinear, hence $\rho(t'(h))=t'(h_{(1)})\otimes h_{(2)}$, and $\rho(\psi(m\otimes h))=t'(h_{(1)})(m\otimes_B 1)\otimes h_{(2)}$. Then we have

$$
\begin{aligned}
(\kappa\circ\psi)(m\otimes h) &= (w'(\psi(m\otimes h)_{[1]}))(\psi(m\otimes h)_{[0]})\\
&= (w'(h_{(2)}))\big(t'(h_{(1)})(m\otimes_B 1)\big)\\
&= ((w'\star t')(h))(m\otimes_B 1)=\beta_{21}(w'\star t')(m\otimes h).
\end{aligned}
$$

(20) is equivalent to $\overline{\beta}_{11}(\varphi\circ\psi)=u'\star t'$.

First observe that

$$
\begin{aligned}
(\overline{\beta}_{11}(\varphi\circ\psi)(h))(m\otimes_B a) &= ((M\otimes\varepsilon)\circ\varphi\circ\psi)(m\otimes h)\otimes_B a\\
&= (\phi\circ\psi)(m\otimes h)\otimes_B a.
\end{aligned}
$$

Now write $\psi(m\otimes h)=\sum_j m_j\otimes_N a_j$. Since ψ is right H-colinear, we have

$$
\psi(m\otimes h_{(1)})\otimes h_{(2)}=\sum_j (m_j\otimes_N a_{j[0]})\otimes_N a_{j[1]}. \tag{24}
$$

Then we compute

$$
\begin{aligned}
\big((u'\star t')(h)\big)(m\otimes_B a) &= \big(u'(h_{(2)})\circ t'(h_{(1)})\big)(m\otimes_B a)\\
&= u'(h_{(2)})\big(\psi(m\otimes h_{(1)})a\big)\\
&\overset{(24)}{=} \sum_j u'(a_{j[1]})\big(\psi(m_j\otimes_N a_{j[0]})a\big)\\
&= \sum_{i,j}\phi\big(m_j\otimes_B l_i(\overline{S}(a_{j[1]}))\big)\otimes_B r_i(\overline{S}(a_{j[1]}))a_{j[0]}a\\
&\overset{(7)}{=} \sum_j \phi(m_j\otimes_B a_j)\otimes_B a=(\phi\circ\psi)(m\otimes h)\otimes_B a.
\end{aligned}
$$

(21) is equivalent to $\beta_{12}(u'\star w')=\varphi\circ\kappa$.

We apply Lemma 3.8 and write

$$
\kappa(m\otimes_B a)=w'(a_{[1]})(m\otimes_B a_{[0]})=\sum_j m_j\otimes_B a_j.
$$

Since κ is right H-colinear, we have

$$
\kappa(m\otimes_B a_{[0]})\otimes a_{[1]}=\sum_j (m_j\otimes_B a_{j[0]})\otimes a_{j[1]}. \tag{25}
$$

Recall from Lemma 3.6 that $\phi(m\otimes_B a)\otimes_B 1=u'(a_{[1]})(m\otimes_B a_{[0]})$. Then

$$
((M\otimes\varepsilon)\circ\varphi\circ\kappa)(m\otimes_B a)\otimes_B 1=(\phi\circ\kappa)(m\otimes_B a)\otimes_B 1
$$

$$
\begin{aligned}
&= \sum_j u'(a_{j[1]})(m \otimes_B a_{j[0]}) \\
&\overset{(25)}{=} u'(a_{[1]})(\kappa(m \otimes_B a_{[0]})) \\
&= \big(u'(a_{[2]}) \circ w'(a_{[1]})\big)(m \otimes_B a_{[0]}) \\
&= \big((u' \star w')(a_{[1]})\big)(m \otimes_B a_{[0]}) \\
&= \big((M \otimes \varepsilon) \circ \beta_{12}(u' \star w')\big)(m \otimes_B a).
\end{aligned}
$$

It follows that $\overline{\delta}_1(\varphi\circ\kappa) = (M\otimes\varepsilon)\circ\varphi\circ\kappa = (M\otimes\varepsilon)\circ\beta_{12}(u'\star w') = \overline{\delta}_1(\beta_{12}(u'\star w'))$, and then $\varphi \circ \kappa = \beta_{12}(u' \star w')$.

(22) is equivalent to $v * t = \overline{\alpha}_{12}(\theta \circ \varphi)$. Recall from (13) that

$$(t(h))(m \otimes_B a) = \sum_i \phi(m \otimes l_i(h)) \otimes_B r_i(h)a,$$

and from Lemma 3.3 that

$$(v(h))(m \otimes_B a) = (v'(S(h)))(m \otimes_B a) = \Theta(m \otimes S(h)) \otimes_B a.$$

Then we compute

$$
\begin{aligned}
((v * t)(h))(m \otimes_B a) &= \big(v(h_{(1)} \circ v(h_{(2)})\big)(m \otimes_B a) \\
&= v(h_{(1)})\big(\sum_i \phi(m \otimes l_i(h_{(2)})) \otimes_B r_i(h_{(2)})a\big) \\
&= \sum_i \Theta\big(\phi(m \otimes l_i(h_{(2)})) \otimes S(h_{(1)})\big) \otimes_B r_i(h_{(2)})a \\
&\overset{(4)}{=} \sum_i \Theta\big(\phi(m \otimes l_i(h)_{[0]}) \otimes l_i(h)_{[1]}\big) \otimes_B r_i(h)a \\
&= \sum_i \Theta(\varphi(m \otimes l_i(h)) \otimes_B r_i(h)a \\
&= \sum_i ((M \otimes \varepsilon) \circ \theta \circ \varphi)(m \otimes l_i(h)) \otimes_B r_i(h)a \\
&\overset{(13)}{=} (\alpha_{12}^{-1}(\theta \circ \varphi))(m \otimes_B a).
\end{aligned}
$$

Finally, (23) is equivalent to $\beta_{22}(t' \star u') = \psi \circ \varphi$. From Lemma 3.5, we have that $\psi(m \otimes h) = t'(h)(m \otimes_B 1)$, and from Lemma 3.6 that $\phi(m \otimes_B a) \otimes_B 1 = u'(a_{[1]})(m \otimes_B a_{[0]})$, hence

$$
\begin{aligned}
(\psi \circ \varphi)(m \otimes_B a) &= \psi\big(\phi(m \otimes_B a_{[0]}) \otimes a_{[1]}\big) \\
&= t'(a_{[1]})\big(\phi(m \otimes_B a_{[0]}) \otimes_B 1\big) \\
&= (t'(a_{[2]}) \circ u'(a_{[1]}))(m \otimes_B a_{[0]}) \\
&= ((t' \star u')(a_{[1]}))(m \otimes_B a_{[0]}) \\
&= (\beta_{22}(t' \star u'))(m \otimes_B a).
\end{aligned}
$$

□

4. THE LEFT-RIGHT CASE

Assume that H is projective as a k-module. Assume that A is a left faithfully flat H-Galois extension of B, that is, A satisfies conditions (4) and (5) of Theorem 1.1. A left A-linear map between left=right (A,H)-modules is called rational if there exists a (unique) $f_{[0]}\otimes f_{[1]}\in {}_A\mathrm{Hom}(P,Q)\otimes H$ such that $\rho(f(p))=f_{[0]}(p_{[0]})\otimes p_{[1]}f_{[1]}$. ${}_A\mathrm{HOM}(P,Q)$, the submodule of rational maps is a right H-comodule and ${}_A\mathrm{END}(P)^{\mathrm{op}}$ is a right H-comodule algebra.

Now take $M\in {}_B\mathcal{M}$, and let $E={}_A\mathrm{END}(A\otimes_B M)^{\mathrm{op}}$. Then $F=E^{\mathrm{co}H}={}_A\mathrm{End}^H(A\otimes_B M)^{\mathrm{op}}\cong {}_B\mathrm{End}(M)^{\mathrm{op}}$. Let $\mathcal{E}_M$ be the full subcategory of ${}_B\mathcal{M}^H$ with objects $B\otimes H$ and $A\otimes_B M$.

THEOREM 4.1. *With notation and assumptions as above, we have a duality* $\alpha:\ \mathcal{C}_E\to\mathcal{E}_M$.

Proof. Let $\alpha(\mathbf{1})=M\otimes H$ and $\alpha(\mathbf{2})=A\otimes_B M$. Below we present the descriptions of the maps $\alpha_{ji}:\ \mathcal{C}_E(\mathbf{i},\mathbf{j})\to\mathcal{D}_M(\mathbf{j},\mathbf{i})$ and their inverses $\overline{\alpha}_{ji}$. All the other verifications are similar to corresponding arguments in the proof of Theorem 3.1 and are left to the reader. Observe that we have two natural isomorphisms

$$\delta_1:\ {}_B\mathrm{Hom}(A\otimes_B M,M)\to {}_B\mathrm{Hom}^H(A\otimes_B M,M\otimes H);$$
$$\delta_2:\ {}_B\mathrm{Hom}(M\otimes H,M)\to {}_B\mathrm{End}^H(M\otimes H)$$

defined as follows:

$$\delta_1(\phi)(a\otimes_B m)=\phi(a_{[0]}\otimes_B m)\otimes a_{[1]}\ ;\ \overline{\delta}_1(\varphi)=(M\otimes\varepsilon)\circ\varphi;$$
$$\delta_2(\Theta)(m\otimes h)=\Theta(m\otimes h_{(1)})\otimes h_{(2)}\ ;\ \overline{\delta}_2(\theta)=(M\otimes\varepsilon)\circ\theta.$$

We have an isomorphism

$$\tilde{\alpha}_{11}:\ \mathcal{C}_E(\mathbf{1},\mathbf{1})=\mathrm{Hom}(H,E^{\mathrm{co}H})\to {}_B\mathrm{Hom}(M\otimes H,M),$$

given by the formulas

$$1\otimes_B\tilde{\alpha}_{11}(v)(m\otimes h)=v(h)(1\otimes_B m);$$
$$\hat{\alpha}_{11}(\Theta)(h)(a\otimes_B m)=a\otimes_B\Theta(m\otimes h).$$

We then define $\alpha_{11}=\beta_2\circ\tilde{\alpha}_{\mathbf{11}}$.

The isomorphism

$$\alpha_{12}:\ \mathcal{C}_E(\mathbf{2},\mathbf{1})=\mathrm{Hom}^H(H,E)\to {}_B\mathrm{Hom}^H(M\otimes H,A\otimes_B M)$$

is given by the formulas

$$\alpha_{12}(t)(m\otimes h)=t(h)(1\otimes_B m)\ ;\ (\overline{\alpha}_{12}(\psi)(h))(a\otimes_B m)=a\psi(m\otimes h).$$

We have an isomorphism

$$\tilde{\alpha}_{21}:\ \mathcal{C}_E(\mathbf{1},\mathbf{2})\to {}_B\mathrm{Hom}(A\otimes_B M,M),$$

given by the formulas

$$1\otimes_B\tilde{\alpha}_{21}(u)(a\otimes_B m)=u(a_{[1]})(a_{[0]}\otimes_B m);$$

$$\big(\hat{\alpha}_{21}(\phi)(h)\big)(a\otimes_B m)=\sum_i al_i(h)\otimes_B \phi(r_i(h)\otimes_B m).$$

We then define $\alpha_{21}=\beta_1\circ\tilde{\alpha}_{\mathbf{21}}$.

Finally, the isomorphism

$$\alpha_{22}:\ \mathcal{C}_E(\mathbf{2},\mathbf{2})\to {}_B\mathrm{End}^H(A\otimes_B M)$$

is given by the formulas

$$\alpha_{22}(w)(a\otimes_B m)=w(a_{[1]})(a_{[0]}\otimes_B m);$$
$$(\overline{\alpha}_{22}(\kappa))(h)(a\otimes_B m)=\sum_i al_i(h)\kappa(r_i(h)\otimes_B m).$$

□

5. CLEFT EXTENSIONS

Recall that a right H-comodule algebra A is called cleft if there exists a convolution invertible $t\in\mathrm{Hom}^H(H,A)$. This means precisely that **1** and **2** are isomorphic objects in $\mathcal{C}_A$.

There is a Structure Theorem for cleft extensions, see [6] or [9, Theorem 7.2.2]: cleft extensions are precisely the crossed product. We will present a proof of this Theorem, based on the duality from Theorem 4.1. First let us recall the precise definition of a crossed product, following [9, Sec. 7.1].

Let H be a Hopf algebra measuring an algebra B: this means that we have a map $\omega:\ H\otimes B\to B$, $\omega(h\otimes b)=h\cdot b$ such that $h\cdot 1=\varepsilon(h)1$ and $h\cdot(bc)=(h_{(1)}\cdot b)(h_{(2)}\cdot c)$, for all $h\in H$ and $b,c\in B$. Let $\sigma:\ H\otimes H\to B$ be a map with convolution inverse $\overline{\sigma}$. $A\#_\sigma H$ is $A\#H$ with multiplication

$$(b\#h)(c\#k)=b(h_{(1)}\cdot c)\sigma(h_{(2)}\otimes k_{(1)})\#h_{(3)}k_{(2)}. \tag{26}$$

The following result originates from [1, 6], see also [9, Lemma 7.1.2]. The proof is straightforward.

Proposition 5.1. *With notation as above, $B\#_\sigma H$ is an associative algebra with unit $1\#1$ if and only if the following conditions hold:*

1) *B is a twisted H-module, this means that $1\cdot b=b$, for all $b\in B$, and*

$$h\cdot(k\cdot b)=\sigma(h_{(1)}\otimes k_{(1)})((h_{(2)}k_{(2)})\cdot b)\overline{\sigma}(h_{(3)}\otimes k_{(3)}), \tag{27}$$

for all $h,k\in H$ and $b\in B$;

2) *σ is a normalized cocycle; this means that $\sigma(h\otimes 1)=\sigma(1\otimes h)=\varepsilon(h)1$ and*

$$\big(h_{(1)}\cdot\sigma(k_{(1)}\otimes l_{(1)})\big)\sigma(h_{(2)}\otimes k_{(2)}l_{(2)})=\sigma(h_{(1)}\otimes k_{(1)})\sigma(h_{(2)}k_{(2)}\otimes l), \tag{28}$$

for all $h,k,l\in H$. Then $B\#_\sigma H$ is called a crossed product; it is an H-comodule algebra, with coaction induced by the comultiplication on H.

Now we present the Structure Theorem for cleft H-comodule algebras. But first we make the following remark. Assume that $t \in \mathrm{Hom}^H(H, A)$ has convolution inverse u. Then $t(1)u(1) = u(1)t(1) = 1$. Then $t' = u(1)t \in \mathrm{Hom}^(H, A)$ has convolution inverse $ut(1)$, and satisfies $t'(1) = 1$. So if A is cleft, then there exists a convolution invertible $t \in \mathrm{Hom}^H(H, A)$ taking the value 1 in 1.

THEOREM 5.2. *Let H be a projective Hopf algebra, A a right H-comodule algebra, and $B = A^{\mathrm{co}H}$. Then the following assertions are equivalent:*

(1) *A is cleft;*
(2) *A is isomorphic to a crossed product $B\#_\sigma H$;*
(3) *A is a faithfully flat left Hopf-Galois extension of B, and A is isomorphic to $B \otimes H$ as a left B-module and a right H-comodule.*

Proof. (1) $\Longrightarrow$ (2). Theorem 4.1 holds under the assumption that A is an H-Galois extension. However, if $M \in {}_B\mathcal{M}$ is such that η'_M is an isomorphism, then we we still have the functor α. This happens in the particular situation where $M = B$. In this case $E = {}_A\mathrm{END}(A \otimes_B B)^{\mathrm{op}} = {}_A\mathrm{END}(A)^{\mathrm{op}} \cong A$, and $F = E^{\mathrm{co}H} = A^{\mathrm{co}H} = B$.

If A is cleft, then there exists a convolution invertible $t \in \mathrm{Hom}^H(H, E)$, with $t(1) = 1$, and then $\alpha_{12}(t) : \ B \otimes H \to A \otimes_B B = A$ is an isomorphism in ${}_B\mathcal{M}^H$. We transport the multiplication on A to $B \otimes H$, and write $B\#_\sigma H$ for $A \otimes H$ with this multiplication. We can easily make this explicit: with notation as in Theorem 4.1, let $\alpha_{12}(t) = \psi$, u the convolution inverse of t, $\tilde{\alpha}_{21}(u) = \phi$ and $\alpha_{21}(u) = \varphi$. Using the formulas in the proof of Theorem 4.1, we find

$$\psi(b \otimes h) = bt(h) \ ; \ \phi(a) = a_{[0]}u(a_{[1]}) \ : \ \varphi(a) = a_{[0]}u(a_{[1]}) \otimes a_{[2]}.$$

Now we transport the multiplication:

$$\begin{aligned}(b\#h)(c\#k) &= \varphi\big(\psi(b\#k)\psi(c\#k)\big) = \varphi(bt(h)ct(k)) \\ &= \ bt(h_{(1)})ct(k_{(1)})u(h_{(2)}k_{(2)}) \otimes h_{(3)}k_{(3)} \\ &= \ bt(h_{(1)})cu(h_{(2)})t(h_{(3)})t(k_{(1)})u(h_{(4)}k_{(2)}) \otimes h_{(5)}k_{(3)}\end{aligned}$$

Now define

$$\omega_t : \ H \otimes B \to B, \ \omega_t(h \otimes b) = t(h_{(1)})bu(h_{(2)}) = h \cdot b, \tag{29}$$

and

$$\sigma : \ H \otimes H \to B, \ \sigma(h \otimes k) = t(h_{(1)})t(k_{(1)})u(h_{(2)}k_{(2)}).$$

Then the multiplication is given by formula (26). The unit of the multiplication is $\varphi(1) = u(1)\#1 = 1\#1$. It is obvious that ω_t measures B and that σ is convolution invertible, with inverse $\overline{\sigma}(h \otimes k) = t(h_{(1)}k_{(1)})u(k_{(2)})u(h_{(2)})$. Straightforward computations show that the conditions of Proposition 5.1 are satisfied, so A is isomorphic to the crossed product $B\#_\sigma H$.

(2) $\Longrightarrow$ (3). Consider a crossed product $A = B\#_\sigma H$, as in Proposition 5.1. Since H is projective, and therefore faithfully flat, as a k-module, A is faithfully

flat as a left and right B-module. Now $A\otimes_B A=(B\otimes H)\otimes_B(B\otimes H)\cong B\otimes H\otimes H$, and then it is easy to see that the canonical map can : $B\otimes H\otimes H\to B\otimes H\otimes H$ is given by the formula

$$\operatorname{can}(a\otimes b\otimes k)=a\sigma(h_{(1)}\otimes k_{(1)})\otimes h_{(2)}k_{(2)}\otimes k_{(3)}.$$

can is bijective, with inverse

$$\operatorname{can}^{-1}(a\otimes b\otimes k)=a\overline{\sigma}(h_{(1)}S(k_{(2)})\otimes k_{(3)})\otimes h_{(2)}S(k_{(1)})\otimes k_{(4)}.$$

Then can' is also bijective, and A is a faithfully flat left and right H-Galois extension, clearly isomorphic to $B\otimes H$ as a left B-module and a right H-comodule.

(2) $\Longrightarrow$ (3). Since A is a faithfully flat left H-Galois extension, we can apply Theorem 4.1. We have an isomorphism $\psi:\ B\otimes H\to A$ in ${}_B\mathcal{M}^H$, and $t=\alpha_{12}(\psi)$ is then a convolution invertible element in $\operatorname{Hom}^H(H,A)$. This shows that A is cleft. □

Remark 5.3. Let $A=B\#_\sigma H$ be a crossed product. From the formulas in Theorem 4.1, we can explicitly compute $t=\alpha_{12}(\psi)$ and $u=\hat{\alpha}_{12}(\phi)$. First, $\psi:\ B\otimes H\to A=B\#_\sigma H$ is the identity map, and then we see easily that $t(h)=1\#h$. In the proof of (2) $\Longrightarrow$ (3), we constructed the inverse of the canonical map, and from this we deduce that

$$\sum_i l_i(h)\otimes r_i(h)=\Big(\overline{\sigma}\big(S(h_{(2)})\otimes h_{(3)}\big)1_B\#S(h_{(1)})\Big)\otimes_B\Big(1_B\#h_{(4)}\Big).$$

Now we have that $\phi=(B\otimes\varepsilon):\ A=B\#_\sigma H\to B$, and then we see that

$$u(h)=\overline{\sigma}\big(S(h_{(2)})\otimes h_{(3)}\big)1_B\#S(h_{(1)}).$$

Of course these formulas are well-known, see for example [9, Prop. 7.2.7].

If $t\in\operatorname{Hom}^H(H,A)$ is an algebra map, then t is convolution invertible, with convolution inverse $t\circ S$. Then the cocycle σ constructed in the proof of Theorem 5.2 is trivial, and (27) reduces to $h\cdot(k\cdot b)=(hk)$, so that B is an H-module algebra. Then A is isomorphic to the smash product $B\#H$. This proves (1) $\Longrightarrow$ (2) in the next theorem.

Theorem 5.4. *Let H be a projective Hopf algebra, A a right H-comodule algebra, and $B=A^{\mathrm{co}H}$. Then the following assertions are equivalent:*

(1) *there exists an algebra map $t\in\operatorname{Hom}^H(H,A)$;*
(2) *A is isomorphic to a smash product $B\#H$.*

Proof. (2) $\Longrightarrow$ (1). The map t constructed in Remark 5.3 is an algebra map. □

Consider the space $\Omega_A=\{t\in\operatorname{Hom}^H(H,A)\mid t\text{ is an algebra map}\}$. We have the following equivalence relation on Ω_A: $t_1\sim t_2$ if and only if there exists $b\in U(B)$ such that $bt_1(h)=t_2(h)b$, for all $h\in H$. We denote $\overline{\Omega}_A=\Omega_A/\sim$. With some extra assumptions, we can give a categorical and cohomological interpretation of Ω_A and $\overline{\Omega}_A$. Throughout the rest of this Section, we will assume that H is cocommutative, B is commutative and A is cleft. In this situation $\mathcal{C}_A(\mathbf{2},\mathbf{2}) = \mathrm{Hom}(H,B)$. For a convolution invertible $t \in \mathrm{Hom}^H(H,A)$, we consider the map ω_t, see (29).

LEMMA 5.5. *ω_t is independent of the choice of t, and makes B into a left H-module algebra.*

Proof. The second statement follows immediately from (28), taking into account that B is commutative. Let $t, t_0 \in \mathrm{Hom}^H(H,A)$ be convolution invertilble, with convolution inverses u and u_0. Using the commutativity of B again, we find

$$u_0(h_{(1)})t(h_{(2)})bu(h_{(3)})t_0(h_{(3)}) = bu_0(h_{(1)})t(h_{(2)})u(h_{(3)})t_0(h_{(3)}) = b.$$

Then

$$\begin{aligned} w_{t_0}(h \otimes b) &= t_0(h_{(1)})bu_0(h_{(2)}) \\ &= t_0(h_{(1)})u_0(h_{(2)})t(h_{(3)})bu(h_{(4)})t_0(h_{(5)})u_0(h_{(6)}) \\ &= t(h_{(1)})bu(h_{(2)}) = w_t(h \otimes b). \end{aligned}$$

□

Since B is a left H-module algebra, we can consider the Sweedler cohomology groups $H^n(H,B)$ with values in B, see [12].

THEOREM 5.6. *Assume that H is cocommutative, B is commutative and H is cleft. Then we have the following subcategory $\mathcal{X}_A$ of $\mathcal{C}_A$. $\mathcal{X}_A$ has two objects* $\mathbf{1}$ *and* $\mathbf{2}$, *and*

$$\begin{aligned} \mathcal{X}_A(\mathbf{1},\mathbf{1}) &= Z^1(H,B); \\ \mathcal{X}_A(\mathbf{2},\mathbf{1}) &= \Omega_A; \\ \mathcal{X}_A(\mathbf{2},\mathbf{2}) &= \{\omega \in \mathrm{Hom}(H,B) \mid \omega \circ S \in Z^1(H,B)\}; \\ \mathcal{X}_A(\mathbf{1},\mathbf{2}) &= \{t \circ S \mid t \in \Omega\}. \end{aligned}$$

Proof. Recall that a convolution invertible $v : H \to B$ is a 1-cocycle in $Z^1(H,B)$ if $v(hk) = (h_{(1)} \cdot v(k))v(h_{(2)})$, for all $h,k \in H$. A convolution invertible $w : H \to B$ lies in $\mathcal{X}_A(\mathbf{2},\mathbf{2})$ if $w(hk) = (S(k_{(1)}) \cdot w(h))w(h_{(2)})$, for all $h,k \in H$. It is well-known that $\mathcal{X}_A(\mathbf{1},\mathbf{1}) = Z^1(H,B)$ and $\mathcal{X}_A(\mathbf{2},\mathbf{2})$ are groups. Take $v \in Z^1(H,A)$, $w = v \circ S \in \mathcal{X}_A(\mathbf{2},\mathbf{2})$, $t,t' \in \Omega_A$, $u = t \circ S, u' = t' \circ S \in \mathcal{X}_A(\mathbf{1},\mathbf{2})$.

1) $t * u_1 \in Z^1(H,B)$: for all $h,k \in H$, we have

$$\begin{aligned} (t_1 * u)(hk) &= t(h_{(1)})t(k_{(1)})u_1(k_{(2)})u_1(h_{(2)}) \\ &= t(h_{(1)})(t * u_1)(k)u(h_{(2)})t(h_{(3)})u_1(h_{(4)}) \\ &= (h_{(1)} \cdot (t * u_1)(k))(t * u)(k). \end{aligned}$$

2) $v * t \in \Omega_A$: for all $h,k \in H$, we have

$$(v * t)(hk) = (h_{(1)} \cdot v(k_{(1)}))v(h_{(2)})t(h_{(3)}t(k_{(2)})$$

$$
\begin{aligned}
&= t(h_{(1)})v(k_{(1)})u(h_{(2)})v(h_{(3)})t(h_{(4)}t(k_{(2)})\\
&= t(h_{(1)})u(h_{(2)})v(h_{(3)})t(h_{(4)}v(k_{(1)})t(k_{(2)})\\
&\qquad (B \text{ is commutative})\\
&= (v*t)(h)(v*t)(k).
\end{aligned}
$$

3) $t*w\in\Omega_A$: for all $h,k\in H$, we have

$$
\begin{aligned}
(t*w)(hk) &= t(h_{(1)})t(k_{(1)})(S(k_{(2)})\cdot w(h_{(2)}))w(k_{(3)})\\
&= t(h_{(1)})t(k_{(1)})u(k_{(2)})w(h_{(2)})t(k_{(3)})w(k_{(3)})\\
&= (t*w)(h)(t*w)(k).
\end{aligned}
$$

4) We know from 1) that $t*u_1\in Z^1(H,B)$, hence $(t*u_1)\circ S = u*t_1\in \mathcal{X}_A(\mathbf{2},\mathbf{2})$.

5) We know from 2) that $v*t\in\Omega_A$, hence $(v*t)\circ S = w*u\in\mathcal{X}_A(\mathbf{1},\mathbf{2})$.

6) We know from 3) that $t*w\in\Omega_A$, hence $(t*w)\circ S = u*v\in\mathcal{X}_A(\mathbf{1},\mathbf{2})$. $\square$

Obviously $\mathcal{X}_A$ is a groupoid: every morphism in $\mathcal{X}_A$ is invertible. Assume now that $\Omega_A\neq\emptyset$, and fix $t_0\in\Omega_A$. Then the map $F:\ Z^1(H,B)\to\Omega_A$, $F(v)=v*t_0$ is a bijection. The inverse is given by $F^{-1}(t)=t*u_0$, with $u_0=t_0\circ S$.

Proposition 5.7. *F sends equivalence classes in $Z^1(H,B)$ to equivalence classes in Ω_A, and a similar property holds for F^{-1}. Hence F induces a bijection $H^1(H,B)\to\overline{\Omega}_A$.*

Proof. For each invertible $b\in B$, we have a 1-cocycle $f_b:\ H\to B$, $f_b(h)=(h\cdot b)b^{-1}$. Then $B^1(H,B)=\{f_b\mid b\in U(B)$, and $H^1(H,B)=Z^1(H,B)/B^1(H,B)$. First assume that $v\sim v_1$ in $Z^1(H,B)$. Then there exist $b\in U(B)$ such that $v=f_b*v_1$. Let $F(v)=t$, $F(v_1)=t_1$, then $t=v*t_0=f_b*v_1*t_0=f_b*t_1$ and

$$
t(h)=b^{-1}(h_{(1)}\cdot b)t_1(h_{(2)})=b^{-1}t_1(h_{(1)})bu_1(h_{(2)})t_{(1)}(h_{(3)})=b^{-1}t_1(h)b,
$$

for all $h\in H$, so that $t\sim t_1$. Conversely, if $t\sim t_1$, then there exists $b\in U(B)$ such that $t(h)=b^{-1}t_1(h)b$, for all $h\in H$, and

$$
\begin{aligned}
(t*u_0)(h) &= b^{-1}t_1(h_{(1)})bu_0(h_{(2)})=b^{-1}t_1(h_{(1)})bu(h_{(2)})t(h_{(3)})u_0(h_{(4)})\\
&= b^{-1}(h_{(1)}\cdot b)(t_1*u_0)(h_{(2)})=(f_b*t_1*u_0)(h),
\end{aligned}
$$

for all $h\in H$, and then $t*u_0$ is cohomologous to t_1*u_0. $\square$

6. STABLE MODULES AND THE MILITARU-ŞTEFAN LIFTING THEOREM

We return to the setting of Section 3: A is a right faithfully flat H-Galois extension, B is the subalgebra of coinvariants, and M is a right B-module. Recall from [11] that M is called H-stable if $M\otimes H$ and $M\otimes_B A$ are isomorphic as right B-modules and right H-comodules. From Theorem 3.1, we immediately obtain the following result, originally due to Schneider [11] in the case where H is finitely generated and projective, and to Militaru and Ştefan, [8, Lemma 3.2] in the general case.

Proposition 6.1. *$M \in \mathcal{M}_B$ is H-stable if and only if $E = \mathrm{END}_A(M \otimes BA)$ is cleft, that is, there exists an H-colinear convolution invertible $t: H \to E$.*

As we have seen in Section 5, an H-colinear algebra map is convolution invertible. Militaru and Ştefan proved that the existence of an H-colinear algebra map $t: H \to E$ is equivalent to the existence of an associative action of A and M extending the right B-action. This can also be derived from Theorem 3.1, which is what we will now discuss. We fix the following notation: $\phi: M \otimes_B A \to A$ is a right B-linear map, $\varphi = \delta_1(\phi)$, $\hat{\beta}_{12}(\phi) = u'$, $t = u \circ S = \hat{\alpha}_{12}(\phi)$. We also write $\phi(m \otimes_B a) = m \cdot a$. From Lemma 3.6, we recall the following formulas (see (12-14):

$$m \cdot a \otimes_B 1 = u'(a_{[1]})(m \otimes_B a_{[0]}); \tag{30}$$

$$t(h)(m \otimes_B a) = \sum_i \phi(m \otimes_B l_i(h)) \otimes_B r_i(h) = \sum_i m \cdot l_i(h) \otimes_B r_i(h). \tag{31}$$

We then immediately have the following result:

Proposition 6.2. *With notation as above, the following assertions are equivalent:*

(1) *$t(1) = 1$;*
(2) *$u'(1) = 1$;*
(3) *$m \cdot 1 = 1$.*

Proof. (1) $\Longrightarrow$ (2) is obvious. (2) $\Longrightarrow$ (3) follows immediately from (30), and (3) $\Longrightarrow$ (1) follows from (31). $\square$

Proposition 6.3. *With notation as above, the following assertions are equivalent:*

(1) *t is multiplicative;*
(2) *u is anti-multiplicative;*
(3) *the right A-action on M defined by ϕ is associative.*

Proof. (1) $\Longrightarrow$ (2) is obvious.
(2) $\Longrightarrow$ (3). For all $m \in M$ and $a, b \in A$, we have

$$\begin{aligned}
(m \cdot (ab)) \otimes_B 1 &\overset{(30)}{=} u'(a_{[1]}b_{[1]})(m \otimes_B a_{[0]}b_{[0]}) \\
&= \big((u'(b_{[1]}) \circ u'(a_{[1]})\big)(m \otimes_B a_{[0]}b_{[0]}) \\
&= u'(b_{[1]})\big(u'(a_{[1]})(m \otimes_B a_{[0]})b_{[0]}\big) \\
&\overset{(30)}{=} u'(b_{[1]})(m \cdot a \otimes_B b_{[0]} \overset{(30)}{=} (m \cdot a) \cdot b.
\end{aligned}$$

(3) $\Longrightarrow$ (1). For all $h, k \in H$, $m \in M$ and $a \in A$, we have

$$t(hk)(m \otimes_B a) \overset{(31)}{=} \sum_i m \cdot l_i(hk) \otimes_B r_i(hk)$$

$$\begin{aligned}
&\overset{(8)}{=} \sum_{i,j} m\cdot(l_i(k)l_j(h))\otimes_B r_j(h)r_i(k)\\
&= \sum_{i,j}(m\cdot l_i(k))\cdot l_j(h)\otimes_B r_j(h)r_i(k)\\
&\overset{(31)}{=} \sum_i t(h)(m\cdot l_i(k)\otimes_B r_i(k))\\
&\overset{(31)}{=} (t(h)\circ t(k))(m\otimes_B a).
\end{aligned}$$

□

Combining these results, we obtain the Militaru-Ştefan lifting Theorem, see [8, Theorem 2.3].

THEOREM 6.4. *With notation as above, the following are equivalent:*

(1) *t is an algebra map;*
(2) *u is an anti-algebra map;*
(3) *ϕ makes M into a right B-module.*

Now consider the set Λ_M consisting of all right B-linear maps $\phi: M\otimes_B A\to M$ defining a right A-module structure on M. It follows from Theorem 6.4 that $\hat\alpha_{12}: \Lambda_M\to\Omega_E$ is a bijection. $\phi_1,\phi_2\in\Lambda_M$ are called equivalent if the resulting right A-modules M_1 and M_2 are isomorphic. Let $\overline{\Lambda}$ be the quotient set.

PROPOSITION 6.5. [8, Theorem 2.6] *Let $\phi_1,\phi_2\in\Lambda_M$, and $t_1=\hat\alpha_{12}(\phi_1)$, $t_2=\hat\alpha_{12}(\phi_2)$ the corresponding H-colinear algebra maps $H\to E$. Then $\phi_1\sim\phi_2$ if and only if $t_1\sim t_2$. Consequently $\overline{\Omega}_E\cong\overline{\Lambda}$ classifies the isomorphism classes of right A-module structures on M extending the right B-action on M.*

Proof. Let $M_i=M$ with right A-action $m\cdot_i a=\phi_i(m\otimes_B a)$, and $u'_i=t_i\circ S^{-1}$ Recall from Section 5 that $t_1\sim t_2$ if and only if there exists an invertbile $f\in\mathrm{End}_B(M)\cong E^{\mathrm{co}H}$ such that

$$t_1(h)\circ(f\otimes_B A)=(f\otimes_B A)\circ t_2(h), \tag{32}$$

or, equivalently,

$$u'_1(h)\circ(f\otimes_B A)=(f\otimes_B A)\circ u'_2(h), \tag{33}$$

$\phi_1\sim\phi_2$ if and only if there exists an invertible $f\in\mathrm{End}_B(M)$ such that $f(m\cdot_2 a)=f(m)\cdot_1 a$, for all $m\in M$ and $a\in A$.

If $t_1\sim t_2$ then

$$\begin{aligned}
f(m\cdot_2 a)\otimes_B 1 &\overset{(12)}{=} ((f\otimes_B A)\circ u'_2(a_{[1]}))(m\otimes_B a_{[0]}\\
&\overset{(32)}{=} (u'_1(a_{[1]})\circ(f\otimes_B A))(m\otimes_B a_{[0]}=f(m)\cdot_1 a\otimes_B 1,
\end{aligned}$$

and it follows that $\phi_1\sim\phi_2$. Conversely, if $\phi_1\sim\phi_2$, then

$$((f\otimes_B A)\circ t_2(h))(m\otimes_B a)\overset{(14)}{=}\sum_i f(m\cdot_2 l_i(h))\otimes_B r_i(h)$$

$$\stackrel{(33)}{=} \sum_i f(m) \cdot_1 l_i(h) \otimes_B r_i(h) = (t_1(h) \circ (f \otimes_B A))(m \otimes_B a),$$

and it follows that $t_1 \sim t_2$. □

If H is cocommutative, $\mathrm{End}_B(M)$ is commutative and $\Omega_E \neq \emptyset$, then we can apply Proposition 5.7, and we obtain a cohomological description of $\overline{\Omega}_E$, namely $\overline{\Omega}_E \cong \overline{\Lambda}_M \cong H^1(H, \mathrm{End}_B(M))$. This result is one of the key arguments in [2].

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