MATHEMATICA, Tome 52 (75), N° 2, 2010, pp. 113–120

LAGRANGE'S EQUILATERAL TRIANGULAR CONFIGURATION: COMPUTATION AND ANALYSIS OF THE RIEMANNIAN CURVATURES

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Abstract. In this paper we study the Riemannian curvatures for the triangular equilateral configuration of Lagrange in the case of the planar three-body problem. The curvature tensor is computed using symbolic computation systems and a result regarding the sign of sectional curvatures is proven.

MSC 2010. 70F07, 57R18.

Key words. Three body problem, Lagrange configuration, Riemannian curvature.

1. INTRODUCTION

The interest in computing the Riemannian curvatures for a conservative dynamical system and its applications are given in [4], [5], [6], [16], [17], [18]. The main ideas are the following:

- For any fixed value of the constant of energy, h, the trajectories of a conservative dynamical system are the geodesics of the n-dimensional configuration space, M^n, provided with the Riemannian Maupertuis' metric.

- Using the symbolic computation program Maple, we compute the Riemannian curvatures for any point Q of M^n.

- A discussion on the sign of the Riemannian curvatures leads to conclusions concerning the behavior (convergence/divergence) of geodesics and, correspondingly, of the trajectories of the configuration space.

We applied this method in the study of the homothetic triangular configurations of Lagrange.

2. THE MAUPERTUIS' METRIC

From Jacobi's form of Maupertuis' Least Action Principle, we know (see [1], [2], [3], [11], [16]) that for any fixed value of the constant of energy, h, the trajectories of a conservative dynamical system are the geodesics of the configuration space M^n, provided with the Riemannian Maupertuis metric

$$\mathrm{d}s^2 = 2(U+h)\mathrm{d}s_0^2, \tag{1}$$

where $\mathrm{d}s_0^2 = 2T\mathrm{d}t^2$, U is the force function ($-U$ being the potential energy) and T is the kinetic energy.

In what follows we shall use Einstein's summation convention, if not stated otherwise. We note that we may also write $\mathrm{d}s_0^2$ as:

$$\mathrm{d}s_0^2 = g^0_{ij}(q)\mathrm{d}q^i\mathrm{d}q^j, \tag{2}$$

where g^0_{ij} are the covariant components of the metric tensor of the space and $q = (q^1, q^2, q^3, \dots, q^n)$ the generalized co-ordinates.

3. COMPUTATION OF THE RIEMANN TENSOR AND RIEMANNIAN CURVATURES

Using the metric tensor (1), with Maple 12 we performed computations of the Riemann tensor and Riemannian/sectional curvatures of the configuration space. We passed through the following stages:

1) Computation of the Christoffel symbols of the second kind, Γ^k_{ij}:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial q^i} + \frac{\partial g_{il}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^1}\right). \tag{3}$$

2) Computation of the Riemann tensor (or the covariant curvature tensor) R_{ijkl}:

$$R_{ijkl} = g_{lr}R^r_{ijk}, \tag{4}$$

with

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial q^j} - \frac{\partial \Gamma^l_{ik}}{\partial q^i} + \Gamma^r_{ik}\Gamma^l_{rj} - \Gamma^r_{jk}\Gamma^l_{ri}.$$

3) Computation of the Riemannian curvatures (or sectional curvatures):

$$K(\pi) = K(u,v) = \frac{R_{ijkl}\cdot u^i \cdot v^j \cdot u^k \cdot v^l}{(g_{jl}\cdot g_{ik} - g_{jk}\cdot g_{il})\cdot u^i \cdot v^j \cdot u^k \cdot v^l}, \tag{5}$$

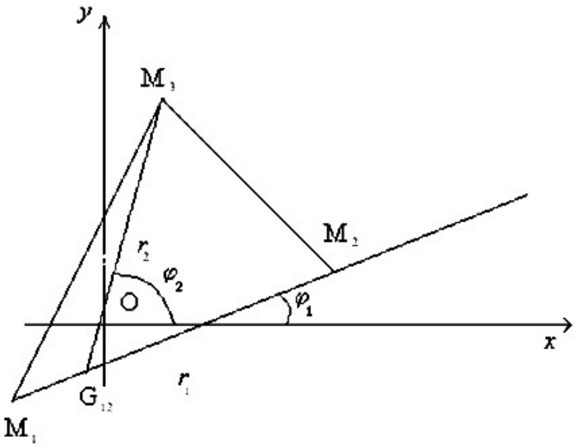
where (u^i) and (v^i) are respectively the components of two linear independent vectors u and v of the tangent space T_QM at $Q \in M$. The vectors u, v determine the 2-dimensional sectional plane, π. If $(v_1, v_2, \dots, v_n)$ is a basis of the tangent space T_QM, we define the "principal sectional curvatures" by $K_{ij} \equiv K(v_i, v_j)$, for $i \neq j$. Note that if the basis of T_QM is orthonormal, we have $K_{ij} = R_{ijij}$. We also remark that in the two dimensional case $K(\pi)$ is the Gauss curvature of a surface spanned by a family of geodesics passing through a same point.

The behavior of two close geodesics is related to the sign of the Riemannian curvatures. A positive Riemannian curvature implies the convergence of neighboring geodesics and a negative Riemannian curvature implies their divergence (see [2], [4], [14], [15], [16]). As we saw, using Maupertuis' metric, the geodesics are the trajectories of the configuration space, so the convergence or divergence of geodesics may be translated in terms of behavior of the trajectories of the dynamical system in the configuration space. Complete proofs of the above statements can be found in [5], [16], [19].

4. THE PLANAR THREE BODY PROBLEM

Let us consider three points of masses m_1, m_2 and m_3 moving in a plane under the mutual Newtonian attraction.

In our study we used Jacobi's co-ordinates, $(r_1, \varphi_1, r_2, \varphi_2)$ described below:



G_{12} is the mass center of M_1 and M_2;
r_1 is the distance M_1M_2;
φ_1 is the angle between r_1 and Ox;
r_2 is the distance $G_{12}M_3$;
φ_2 is the angle between r_2 and Ox.

With these co-ordinates, Maupertuis' metric becomes:

$$\mathrm{d}s^2 = (U+h)[\mu_1(\mathrm{d}r_1^2 + r_1^2\mathrm{d}\varphi_1^2) + \mu_2(\mathrm{d}r_2^2 + r_2^d\varphi_2^2)] \tag{6}$$

with

$$\mu_1 = \frac{m_1m_2}{m_1+m_2}, \quad \mu_2 = \frac{m_3(m_1+m_2)}{m_1+m_2+m_3}. \tag{7}$$

Note that, comparing (6) with the general forms described by (1) and (2), the coordinates are

$$q^1 = r_1, \quad q^2 = \varphi_1, \quad q^3 = \varphi_2, \quad q^4 = r_2$$

and the covariant components of Maupertuis' metric in the configuration space of the planar three-body problem are:

$$\begin{array}{l} g_{11} = \mu_1(U+h); \quad g_{22} = \mu_1 r_1^2(U+h); \\ g_{33} = \mu_2(U+h); \quad g_{44} = \mu_2 r_2^2(U+h) \\ g_{ij} = 0, \text{ for } i \neq j. \end{array} \tag{8}$$

The force function U reads as:

$$U = \frac{m_1m_2}{r_1} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}, \tag{9}$$

where

(10) $r_{13}^2 = (\varepsilon_1 r_1)^2 + r_2^2 + 2\varepsilon_1 r_1 r_2 \cos\varphi, \quad r_{23}^2 = (\varepsilon_2 r_1)^2 + r_2^2 - 2\varepsilon_2 r_1 r_2 \cos\varphi$

$$\varepsilon_1 = \frac{m_2}{m_1+m_2}, \quad \varepsilon_2 = \frac{m_1}{m_1+m_2}, \quad \varphi = \varphi_2 - \varphi_1.$$

The kinetic energy, T is given by:

$$2T = \mu_1\left[\left(\frac{\mathrm{d}r_1}{\mathrm{d}t}\right)^2 + r_1^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\right] + \mu_2\left[\left(\frac{\mathrm{d}r_2}{\mathrm{d}t}\right)^2 + r_2^2\left(\frac{\mathrm{d}\varphi_2}{\mathrm{d}t}\right)^2\right]. \tag{11}$$

The configuration space of the planar three-body problem is a 4-dimensional space; the vectors corresponding to the principal sectional 2-planes are:

$$v_1\left(\frac{1}{\sqrt{\mu_1}},0,0,0\right), \quad v_2\left(0,\frac{1}{r_1\sqrt{\mu_1}},0,0\right),$$
$$v_3\left(0,0,\frac{1}{\sqrt{\mu_2}},\right), \quad v_4\left(0,0,0,\frac{1}{r_2\sqrt{\mu_2}}\right),$$

where $\|v_i\| = 1$, $i = 1,\ldots,4$ and $\langle v_i, v_j\rangle = 0$, $i \neq j$.

Using the Maple 12, from (5) we obtained the six principal Riemannian curvatures, $K(v_i, v_j) = K_{ij}$:

(12)

$$K_{12} = \frac{\left[\begin{array}{c} 2\mu_2 r_2^2(U+h)\left(r_1^2\dfrac{\partial^2 U}{\partial r_1^2} + r_1\dfrac{\partial U}{\partial r_1} + \dfrac{\partial^2 U}{\partial \varphi_1^2}\right) - \\ -2\mu_2 r_1^2 r_2^2\left(\dfrac{\partial U}{\partial r_1}\right)^2 - 2\mu_2 r_2^2\left(\dfrac{\partial U}{\partial \varphi_1}\right)^2 + \mu_1 r_1^2 r_2^2\left(\dfrac{\partial U}{\partial r_2}\right)^2 + \mu_1 r_1^2\left(\dfrac{\partial U}{\partial \varphi_2}\right)^2 \end{array}\right]}{4\mu_1\mu_2 r_1^2 r_2^2(U+h)^3}$$

$$K_{13} = \frac{\left[\begin{array}{c} 2r_1^2 r_2^2(U+h)\left(\mu_2\dfrac{\partial^2 U}{\partial r_1^2} + \mu_1\dfrac{\partial^2 U}{\partial r_2^2}\right) - \\ -2\mu_2 r_1^2 r_2^2\left(\dfrac{\partial U}{\partial r_1}\right)^2 + \mu_2 r_2^2\left(\dfrac{\partial U}{\partial \varphi_1}\right)^2 - 2\mu_1 r_1^2 r_2^2\left(\dfrac{\partial U}{\partial r_2}\right)^2 + \mu_1 r_1^2\left(\dfrac{\partial U}{\partial \varphi_2}\right)^2 \end{array}\right]}{4\mu_1\mu_2 r_1^2 r_2^2(U+h)^3}$$

$$K_{14} = \frac{\left[\begin{array}{c} 2r_1^2(U+h)\left(\mu_2 r_2^2\dfrac{\partial^2 U}{\partial r_1^2} + \mu_1 r_2\dfrac{\partial U}{\partial r_2} + \mu_1\dfrac{\partial^2 U}{\partial \varphi_2^2}\right) - \\ -2\mu_2 r_1^2 r_2^2\left(\dfrac{\partial U}{\partial r_1}\right)^2 + \mu_2 r_2^2\left(\dfrac{\partial U}{\partial \varphi_1}\right)^2 + \mu_1 r_1^2 r_2^2\left(\dfrac{\partial U}{\partial r_2}\right)^2 - 2\mu_1 r_1^2\left(\dfrac{\partial U}{\partial \varphi_2}\right)^2 \end{array}\right]}{4\mu_1\mu_2 r_1^2 r_2^2(U+h)^3}$$

$$K_{23} = \frac{\left[\begin{array}{c} 2r_2^2(U+h)\left(\mu_2 r_1\dfrac{\partial U}{\partial r_1} + \mu_2\dfrac{\partial^2 U}{\partial \varphi_1^2} + \mu_1 r_1^2\dfrac{\partial^2 U}{\partial r_2^2}\right) + \\ +\mu_2 r_1^2 r_2^2\left(\dfrac{\partial U}{\partial r_1}\right)^2 - 2\mu_2 r_2^2\left(\dfrac{\partial U}{\partial \varphi_1}\right)^2 - 2\mu_1 r_1^2 r_2^2\left(\dfrac{\partial U}{\partial r_2}\right)^2 + \mu_1 r_1^2\left(\dfrac{\partial U}{\partial \varphi_2}\right)^2 \end{array}\right]}{4\mu_1\mu_2 r_1^2 r_2^2(U+h)^3}$$

$$K_{24} = \frac{\begin{bmatrix} 2(U+h)\left(\mu_2 r_1 r_2^2 \frac{\partial U}{\partial r_1} + \mu_2 r_2^2 \frac{\partial^2 U}{\partial \varphi_1^2} + \mu_1 r_1^2 r_2 \frac{\partial U}{\partial r_2} + \mu_1 r_1^2 \frac{\partial^2 U}{\partial \varphi_2^2}\right) + \\ +\mu_2 r_1^2 r_2^2 \left(\frac{\partial U}{\partial r_1}\right)^2 - 2\mu_2 r_2^2 \left(\frac{\partial U}{\partial \varphi_1}\right) + \mu_1 r_1^2 r_2^2 \left(\frac{\partial U}{\partial r_2}\right)^2 - 2\mu_1 r_1^2 \left(\frac{\partial U}{\partial \varphi_2}\right)^2 \end{bmatrix}}{4\mu_1 \mu_2 r_1^2 r_2^2 (U+h)^3}$$

$$K_{34} = \frac{\begin{bmatrix} 2\mu_1 r_1^2 (U+h)\left(r_2^2 \frac{\partial^2 U}{\partial r_2^2} + r_2 \frac{\partial U}{\partial r_2} + \frac{\partial^2 U}{\partial \varphi_2^2}\right) + \\ +\mu_2 r_1^2 r_2^2 \left(\frac{\partial U}{\partial r_1}\right)^2 + \mu_2 r_2^2 \left(\frac{\partial U}{\partial \varphi_1}\right)^2 - 2\mu_1 r_1^2 r_2^2 \left(\frac{\partial U}{\partial r_2}\right)^2 - 2\mu_1 r_1^2 \left(\frac{\partial U}{\partial \varphi_2}\right)^2 \end{bmatrix}}{4\mu_1 \mu_2 r_1^2 r_2^2 (U+h)^3}.$$

5. LAGRANGE'S EQUILATERAL SOLUTION FOR UNEQUAL MASSES

For the particular homothetic solution of Lagrange's equilateral triangle configuration, we consider $m_1 > m_2 > m_3$ (comparable masses), the gravitation constant equal to 1 and $r_1 = r_{13} = r_{23} = r = r(t)$, t being the time and r the common side of the equilateral triangle $M_1M_2M_3$. Taking into account the shape of the triangle, we also have

$$\varphi = \arctan \frac{\sqrt{3}(m_1+m_2)}{m_1-m_2} \quad \text{and} \quad r_2 = \frac{r\sqrt{m_1^2+m_1m_2+m_2^2}}{m_1+m_2}.$$

The elliptic trajectories of the three points have the same focus, O, the same eccentricity e and the same period P, such that:

$$\frac{P}{2\pi} = \frac{a_1^{3/2}}{v_1^{1/2}} = \frac{a_2^{3/2}}{v_2^{1/2}} = \frac{a_3^{3/2}}{v_3^{1/2}}, \tag{13}$$

where a_i is the semi-major axis of M_i

$$v_1 = \frac{(m_2^2+m_3^2+m_2m_3)^{3/2}}{M^2}, \quad V_2 = \frac{(m_1^2+m_3^2+m_1m_3)^{3/2}}{M^2}$$

$$v_3 = \frac{(m_1^2+m_2^2+m_1m_2)^{3/2}}{M^2}, \quad M = m_1+m_2+m_3.$$

The energy constant verifies:

$$h = -\frac{1}{2}\sum_{i=1}^{3} m_i \cdot \frac{v_i}{a_i}. \tag{14}$$

We have also:

$$|OM_i| = r \cdot \alpha_i,$$

with $\alpha_i = \left[\frac{v_i}{f \cdot M}\right]^{1/3}$, $i = 1, 2, 3$.

Analytical computation led us to the following estimation of r:

$$r_{min} \leq r \leq r_{max}, \tag{15}$$

with

$$r_{min}^2 = \frac{M(1-e)^2\left[\sum_{i=1}^{3} m_i a_i^2\right]}{m_1m_2 + m_2m_3 + m_3m_1} \quad \text{and} \quad r_{max}^2 = \frac{M(1+e)^2\left[\sum_{i=1}^{3} m_i a_i^2\right]}{m_1m_2 + m_2m_3 + m_3m_1}. \tag{16}$$

We denote

$$\xi = hr \quad \text{and} \quad \lambda = m_1m_2 + m_2m_3 + m_3m_1. \tag{17}$$

For the equilateral triangle, from $U + h > 0$, we get the condition:

$$\xi + \lambda > 0. \tag{18}$$

THEOREM 5.1. *For Lagrange's triangular configuration of the planar three body problem, there is always at least one negative principal sectional curvature.*

Proof. From (12), sectional curvatures were computed in the particular case of Lagrange ($r_1 = r_{13} = r_{23}$). As a result of these computations, after simplifying and factoring positive terms, we concluded that each principal Riemannian curvature can be written as a product of a positive function and a function S_i:

$$S_i = -[\xi f_i(m_1, m_2, m_3) + g_i(m_1, m_2, m_3)], \tag{19}$$

where S_1, S_2, S_3, S_4, S_5 and S_6 give respectively the signs of K_{12}, K_{13}, K_{14}, K_{23}, K_{24} and K_{34}.

Specifically, we have:

For S_1:

$$\begin{aligned} f_1(m_1, m_2, m_3) &= 2(m_1 + m_2), \\ g_1(m_1, m_2, m_3) &= 3m_3(m_1^2 + m_1m_2 + m_2^2). \end{aligned}$$

For S_2:

$$\begin{aligned} f_2(m_1, m_2, m_3) &= 7m_2^2m_3 + m_1m_3(7m_1 - 2m_2) + 16m_2^3 \\ &\quad + 23m_1m_2^2 + 23m_1^2m_2 + 16m_1^3, \\ g_2(m_1, m_2, m_3) &= 3(m_2m_3 + m_1m_3 + m_1m_2)[m_2^2m_3 \\ &\quad + m_1m_2(5m_1 - 2m_3) + m_1^2m_3 + 4m_2^3 + 5m_1m_2^2 + 4m_1^3]. \end{aligned}$$

For S_3:

$$\begin{aligned} f_3(m_1,m_2,m_3) &= (m_1+m_2)[m_2^2(17m_1-5m_3)+4m_1m_2m_3 \\ &\quad + m_1^2(17m_2-5m_3)+4m_2^3+4m_1^3], \\ g_3(m_1,m_2,m_3) &= 3[m_2^4m_3(2m_2-m_3)+2m_1m_2^3m_3(8m_1-m_3) \\ &\quad + 2m_1^3m_2m_3(8m_2-m_3)+m_1^4m_3(6m_2-m_3)+6m_1m_3m_2^4 \\ &\quad + 2m_1^5m_3+3m_1^2m_2^4+6m_1^3m_2^3+3m_1^4m_2^2]. \end{aligned}$$

For S_4:

$$\begin{aligned} f_4(m_1,m_2,m_3) &= (m_1+m_2)[13m_2^2m_3+4m_1m_2m_3+13m_1^2m_3 \\ &\quad + 4m_2^3+m_1(2m_1^2-m_2^2)+m_1^2(2m_1-m_2)], \\ g_4(m_1,m_2,m_3) &= 3(3m_2^4m_3^2+8m_1m_2^3m_3^2+8m_1^2m_2^2m_3^2 \\ &\quad + 8m_1^3m_2m_3^2+3m_1^4m_3^2+4m_1m_2^4m_3 \\ &\quad + 2m_1^2m_2^3m_3+2m_1^3m_2^2m_3+4m_1^4m_2m_3 \\ &\quad + 2m_1m_2^5+3m_1^2m_2^4+2m_1^3m_2^3+3m_1^4m_2^2+2m_1^5m_2). \end{aligned}$$

For S_5:

$$\begin{aligned} f_5(m_1,m_2,m_3) &= -[m_2^2(8m_2-m_3)+m_1^2(8m_1-m_3) \\ &\quad + m_1m_2(7m_2-5m_3)+m_1m_2(7m_1-5m_3)], \\ g_5(m_1,m_2,m_3) &= 3(m_2m_3+m_1m_3+m_1m_2)(m_2^2m_3+4m_1m_2m_3 \\ &\quad + m_1^2m_3-2m_2^3-m_1m_2^2-m_1^2m_2-2m_1^3). \end{aligned}$$

For S_6:

$$\begin{aligned} f_6(m_1,m_2,m_3) &= 2(m_1+m_2), \\ g_6(m_1,m_2,m_3) &= 3m_1m_2(m_3+m_2+m_1). \end{aligned}$$

Let us analyze the sign of the above functions, S_i $(i=1,\dots,6)$. From (19), $\xi_i = -\frac{g_i}{f_i}$, $i=1,\dots,6$, therefore $S_i<0 \;\Leftrightarrow\; \xi>\xi_i$ for $i=1,\dots,6$. As for the condition (18), we have $\xi_1>\lambda$, $\xi_2>$, $\xi_3>\lambda$, $\xi_5>\lambda$, while the condition $(m_1>m_2>m_3)$ does not imply $\xi_4>\lambda$, $\xi_6>\lambda$. Furthermore, if $\xi_2<\xi<\xi_5$, then $S_2<0$ and $S_5>0$. Additionally, if $\xi_5>\xi_m=\max\{\xi_1,\xi_2,\xi_3,\xi_4,\xi_6\}$ and $\xi\in[\xi_m,\xi_5]\cap[\lambda,\infty[$ then $S_i<0$, $i=1,\dots,6$.

Consequently, we have proved that there will always be at least one $S_i<0$. Thus, there is at least one negative principal Riemannian curvature. □

The interest of studying the signs of sectional curvatures is justified by the connection between the signs of the curvatures and the stability of trajectories, which will be the subject of a future work.

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Received September 3, 2009
Accepted October 12, 2009

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