MATHEMATICA, Tome 52 (75), N° 2, 2010, pp. 99–111

A SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Making use of the Salagean operator, we define the class $T(n, \alpha, \beta)$. When $n = 1$ and $n = 0$, we obtain, respectively, a new subclass of uniformly convex functions and a corresponding subclass of starlike functions with negative coefficients. In this paper, we obtain distortion theorem, and obtain radii of close-to-convexity, starlikeness and convexity for functions beloning to the class $T(n, \alpha, \beta)$.We consider integral operators associated with functions belonging to the class $T(n, \alpha, \beta)$. We also obtain several results for the modified Hadamard products of functions belonging to the class $T(n, \alpha, \beta)$. Distortion theorem for the fractional calculus (that is, fractional integral and fractional derivative) of functions in the class $T(n, \alpha, \beta)$ is obtained.

MSC 2010. 30C45.
Key words. Salagean operator, analytic function, uniformly convex function, modified Hadamard products, fractional calculus.

1. INTRODUCTION

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the subclasses of S that are, respectively, convex and starlike functions of order α with $0 \le \alpha < 1$. For convenience, we write $K(0) = K$ and $S^*(0) = S^*$ (see, e.g., Srivastava and Owa [17]). Goodman ([2] and [3]) defined the following subclasses of K and S^*.

Definition 1. A function $f(z)$ is *uniformly convex* (*starlike*) in U if $f(z)$ is in $K(S^*)$ and has the property that for every circular γ contained in U, with center ζ also in U, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\zeta)$.

Goodman ([2] and [3]) gave the following two-variable analytic characterizations of these classes, denoted by UCV and UST, respectively.

Theorem 1. *A function $f(z)$ of the form* (1.1) *is in UCV if and only if*

$$\operatorname{Re}\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} \ge 0,\ (z,\zeta) \in U \times U, \tag{1.2}$$

The author is thankful to the referee for his/her comments and suggestions.

and is in UST if and only if

$$\text{Re}\left\{\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\right\}\geq 0,\ (z,\zeta)\in U\times U. \tag{1.3}$$

Ma and Minda [6] and Ronning [11] found independently a more applicable one-variable characterization for UCV.

THEOREM 2. *A function $f(z)$ of the form* (1.1) *is in UCV if and only if*

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geq\left|\frac{zf''(z)}{f'(z)}\right|,\ z\in U. \tag{1.4}$$

We note (see [2]) that Alexander's classical result, $f(z)\in K\Leftrightarrow zf'(z)\in S^*$, does not hold between the classes UCV and UST. Later on, Ronning [12] introduced a new class S_p of starlike functions related to UCV defined by

$$f(z)\in S_p\Leftrightarrow \text{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\geq\left|\frac{zf'(z)}{f(z)}-1\right|,\ z\in U. \tag{1.5}$$

Note that $f(z)\in UCV\Leftrightarrow zf'(z)\in S_p$.

Also in [11], Ronning generalized the classes UCV and S_p by introducing a parameter α in the following way.

DEFINITION 2. A function $f(z)$ of the form (1.1) is in $S_p(\alpha)$ if it satisfies the analytic characterization

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}\geq\left|\frac{zf'(z)}{f(z)}-1\right|,\ \alpha\in R,\ z\in U. \tag{1.7}$$

One says that $f(z)\in UCV(\alpha)$, i.e., f belongs to the class of uniformly convex functions of order α, if and only if $zf'(z)\in S_p(\alpha)$.

For the class $S_p(\alpha)$, we get a domain whose boundary is a parabola with vertex $w=\frac{1+\alpha}{2}$. Note also that $S_p(\alpha)\subset S^*$ for all $-1\leq\alpha<1, S_p(\alpha)\nsubseteq S$ for $\alpha<-1$, and $UCV(\alpha)\subset K$ for $\alpha\geq -1$.

By β-UCV, where $0\leq\beta<\infty$, we denote the class of all β-uniformly convex functions introduced by Kanas and Wisniowska [4]. Recall that a function $f(z)\in S$ is said to be *β-uniformly convex* in U if the image of every circular arc contained in U with center at ζ, where $|\zeta|\leq\beta$, is convex. Note that the class 1-UCV coincides with the class UCV. Moreover, for $\beta=0$ we get the class K. It is known that $f(z)\in\beta$-UCV if and only if it satisfies the following condition

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta\left|\frac{zf''(z)}{f'(z)}\right|,\ z\in U,\ 0\leq\beta<\infty\,. \tag{1.8}$$

We consider the class β-S_p, with $0 \leq \beta < \infty$, of β-starlike functions (see [5]) which are associated with β-uniformly convex functions by the relation

$$f(z) \in \beta\text{-}UCV \Leftrightarrow zf^{'}(z) \in \beta\text{-}S_p. \tag{1.9}$$

Thus, the class β-S_p, with $0 \leq \beta < \infty$, is the subclass of S consisting of functions that satisfy the analytic condition

$$\mathrm{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \beta\left|\frac{zf^{'}(z)}{f(z)} - 1\right|,\ z \in U. \tag{1.10}$$

For a function $f(z)$ in S we define: $D^0 f(z) = f(z)$, $D^1 f(z) = Df(z) = zf^{'}(z)$, and $D^n f(z) = D(D^{n-1}f(z))$ $(n \in \mathbb{N} = \{1, 2, \dots\})$. The differential operator D^n was introduced by Salagean in [14]. It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,\ \text{for all } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{1.14}$$

For $\beta \geq 0$, $-1 \leq \alpha < 1$, and $n \in \mathbb{N}_0$ let $S^n(\alpha, \beta)$ denote the subclass of S consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic condition

$$\mathrm{Re}\left\{\frac{z(D^n f(z))^{'}}{D^n f(z)} - \alpha\right\} > \beta\left|\frac{z(D^n f(z))^{'}}{D^n f(z)} - 1\right|,\ z \in U. \tag{1.15}$$

We note that $S^1(\alpha, \beta) = \beta\text{-}UCV(\alpha)$ and $S^0(\alpha, \beta) = \beta\text{-}S_p(\alpha)$.

We denote by T the subclass of S that consists of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k\ (a_k \geq 0). \tag{1.18}$$

Further, we define the class $T(n, \alpha, \beta) = S^n(\alpha, \beta) \cap T$. The class $T(n, \alpha, \beta)$ was introduced and studied by Rosy and Murugusundaramoorthy in [13]. We also note that $T(0, \alpha, 0) = T^*(\alpha)$ $(0 \leq \alpha < 1)$ and $T(1, \alpha, 0) = C(\alpha)$ $(0 \leq \alpha < 1)$ (Silverman [16]); $T(n, \alpha, 0) = T^*(n, \alpha)$ $(0 \leq \alpha < 1)$ (Hur and Oh [1]).

In order to show our main results we need the following lemma given by Rosy and Murugusundaramoorthy [13].

Lemma 1. *A necessary and sufficient condition for the function $f(z)$ of the form* (1.18) *to be in the class $T(n, \alpha, \beta)$ $(n \in \mathbb{N}_0,\ -1 \leq \alpha < 1,\ \beta \geq 0)$ is that*

$$\sum_{k=2}^{\infty} k^n[k(1+\beta) - (\alpha + \beta)]a_k \leq 1 - \alpha. \tag{1.20}$$

Remark 1. Putting $n = \alpha = 0$ and $\beta = 1$ in Lemma 1, we obtain the result obtained by Ravichandran in [10, Corollary 2].

2. THE GROWTH AND DISTORTION THEOREM

THEOREM 3. *Let $f(z)$ defined by (1.18) be in the class $T(n,\alpha,\beta)$. Then*

$$|z| - \frac{1-\alpha}{2^{n-i}(2-\alpha+\beta)}|z|^2 \le \left|D^i f(z)\right| \le |z| + \frac{1-\alpha}{2^{n-i}(2-\alpha+\beta)}|z|^2, \tag{2.1}$$

where $z \in U$ and $0 \le i \le n$. The bounds are attained for the function

$$f(z) = z - \frac{1-\alpha}{2^n(2-\alpha+\beta)}z^2 \quad (z \in U). \tag{2.2}$$

Proof. Note that $f(z) \in T(n,\alpha,\beta)$ if and only if $D^i f(z) \in T(n-i,\alpha,\beta)$ and that

$$D^i f(z) = z - \sum_{k=2}^{\infty} k^i a_k z^k. \tag{2.3}$$

Using Lemma 1, we know that

$$2^{n-i}(2-\alpha+\beta)\sum_{k=2}^{\infty} k^i a_k \le \sum_{k=2}^{\infty} k^n[k(1+\beta)-(\alpha+\beta)]a_k \le 1-\alpha, \tag{2.4}$$

that is

$$\sum_{k=2}^{\infty} k^i a_k \le \frac{1-\alpha}{2^{n-i}(2-\alpha+\beta)}. \tag{2.5}$$

It follows from (2.3) and (2.5) that

$$\left|D^i f(z)\right| \ge |z| - |z|^2\sum_{k=2}^{\infty} k^i a_k \ge |z| - \frac{1-\alpha}{2^{n-i}(2-\alpha+\beta)}|z|^2 \tag{2.6}$$

and

$$\left|D^i f(z)\right| \le |z| + |z|^2\sum_{k=2}^{\infty} k^i a_k \le |z| + \frac{1-\alpha}{2^{n-i}(2-\alpha+\beta)}|z|^2. \tag{2.7}$$

Finally, we note that the bounds in (2.1) are attained for $f(z)$ defined by

$$D^i f(z) = z - \frac{1-\alpha}{2^{n-i}(2-\alpha+\beta)}z^2 \quad (z \in U). \tag{2.8}$$

This completes the proof of Theorem 3. □

COROLLARY 1. *Let $f(z)$ defined by (1.18) be in the class $T(n,\alpha,\beta)$. Then*

$$|z| - \frac{1-\alpha}{2^n(2-\alpha+\beta)}|z|^2 \le |f(z)| \le |z| + \frac{1-\alpha}{2^n(2-\alpha+\beta)}|z|^2. \tag{2.9}$$

The equalities in (2.9) are attained for the function $f(z)$ given by (2.2).

Proof. Taking $i = 0$ in Theorem 3, we immediately obtain (2.9). □

Corollary 2. *Let $f(z)$ defined by* (1.18) *be in the class $T(n,\alpha,\beta)$. Then*

$$1-\frac{1-\alpha}{2^{n-1}(2-\alpha+\beta)}\left|z\right| \le \left|f'(z)\right| \le 1+\frac{1-\alpha}{2^{n-1}(2-\alpha+\beta)}\left|z\right|. \tag{2.10}$$

The equalities in (2.10) *are attained for the function $f(z)$ given by* (2.2).

Proof. Setting $i=1$ in Theorem 3, and making use of the definition of D^1, we get the conclusion. □

3. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

Theorem 4. *Let the function $f(z)$ defined by* (1.18) *be in the class $T(n,\alpha,\beta)$. Then $f(z)$ is close-to-convex of order ρ $(0\le\rho<1)$ in $|z|<r_1$, where*

$$r_1=r_1(n,\alpha,\beta,\rho)=\inf_{k\ge 2}\left\{\frac{(1-\rho)k^{n-1}[k(1+\beta)-(\alpha+\beta)]}{1-\alpha}\right\}^{\frac{1}{k-1}}. \tag{3.1}$$

The result is sharp, the extremal function $f(z)$ being given by

$$f(z)=z-\frac{(1-\alpha)}{k^n[k(1+\beta)-(\alpha+\beta)]}z^k \quad (k\ge 2,\ n\in\mathbb{N}_0). \tag{3.2}$$

Proof. We must show that $|f'(z)-1|\le 1-\rho$ for $|z|<r_1(n,\alpha,\beta,\rho)$, where $r_1(n,\alpha,\beta,\rho)$ is given by (3.1). Indeed we find from the definition (1.18) that

$$\left|f'(z)-1\right| \le \sum_{k=2}^{\infty} k a_k \left|z\right|^{k-1}.$$

Thus $|f'(z)-1|\le 1-\rho$ if

$$\sum_{k=2}^{\infty}(\frac{k}{1-\rho})a_k\left|z\right|^{k-1}\le 1. \tag{3.3}$$

But, by Lemma 1, (3.3) will be true if

$$\left(\frac{k}{1-\rho})\right)\left|z\right|^{k-1} \le \frac{k^n[k(1+\beta)-(\alpha+\beta)]}{1-\alpha},$$

that is, if

$$|z|\le\left\{\frac{(1-\rho)k^{n-1}[k(1+\beta)-(\alpha+\beta)]}{1-\alpha}\right\}^{\frac{1}{k-1}} \ (k\ge 2). \tag{3.4}$$

Now Theorem 4 follows easily from (3.4). □

Theorem 5. *Let the function $f(z)$ defined by* (1.18) *be in the class $T(n,\alpha,\beta)$. Then the function $f(z)$ is starlike of order ρ $(0\le\rho<1)$ in $|z|<r_2$, where*

$$r_2=r_2(n,\alpha,\beta,\rho)=\inf_{k\ge 2}\left\{\frac{(1-\rho)k^{n}[k(1+\beta)-(\alpha+\beta)]}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{3.5}$$

The result is sharp, with the extremal function $f(z)$ given by (3.2).

Proof. It suffices to show that $\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho$ for $|z| < r_2(n,\alpha,\beta,\rho)$, where $r_2(n,\alpha,\beta,\rho)$ is given by (3.5). Indeed we find, again from (1.18), that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k\,|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k\,|z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho$ if

$$\sum_{k=2}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k\,|z|^{k-1} \le 1. \tag{3.6}$$

But, by Lemma 1, (3.6) will be true if

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1} \le \frac{k^n[k(1+\beta)-(\alpha+\beta)]}{1-\alpha},$$

that is, if

$$|z| \le \left\{\frac{(1-\rho)k^n[k(1+\beta)-(\alpha+\beta)]}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{3.7}$$

Now Theorem 5 follows easily from (3.7). □

COROLLARY 3. *Let the function $f(z)$ defined by* (1.18) *be in the class $T(n,\alpha,\beta)$. Then $f(z)$ is convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = r_3(n,\alpha,\beta,\rho) = \inf_{k\ge 2}\left\{\frac{(1-\rho)k^{n-1}[k(1+\beta)-(\alpha+\beta)]}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{3.8}$$

The result is sharp, with the extremal function $f(z)$ given by (3.2).

4. A FAMILY OF INTEGRAL OPERATORS

In view of Lemma 1, we see that $z-\sum\limits_{k=2}^{\infty}b_kz^k$ is in $T(n,\alpha,\beta)$ as long as $0 \le b_k \le a_k$ for all k. In particular, we have the following result

THEOREM 6. *Let the function $f(z)$ defined by* (1.18) *be in the class $T(n,\alpha,\beta)$ and let $c > -1$ be a real number. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\mathrm{d}t \quad (c > -1) \tag{4.1}$$

also belongs to the class $T(n,\alpha,\beta)$.

Proof. It follows from the representation (4.1) that $F(z) = z-\sum\limits_{k=2}^{\infty}b_kz^k$, where $b_k = \frac{c+1}{c+k}a_k \le a_k$. □

On the other hand, the converse is not true. This leads to a radius of univalence result.

Theorem 7. *Let the function* $F(z) = z - \sum_{k=2}^{\infty} a_k z^k$ $(a_k \geq 0)$ *be in the class* $T(n, \alpha, \beta)$ *and let* $c > -1$ *be a real number. Then the function* $f(z)$ *given by* (4.1) *is univalent in* $|z| < R^*$, *where*

$$R^* = \inf_{k \geq 2} \left\{ \frac{k^{n-1}[k(1+\beta) - (\alpha+\beta)](c+1)}{(1-\alpha)(c+k)} \right\}^{\frac{1}{k-1}}. \tag{4.2}$$

The result is sharp.

Proof. From (4.1), we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} = z - \sum_{k=2}^{\infty} \frac{c+k}{c+1} a_k z^k.$$

In order to obtain the required result, it suffices to show that $|f'(z) - 1| < 1$ whenever $|z| < R^*$, where R^* is given by (4.2). Now

$$\left|f'(z) - 1\right| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k \; |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k \; |z|^{k-1} < 1. \tag{4.3}$$

But Lemma 1 confirms that

$$\sum_{k=2}^{\infty} \frac{k^n[k(1+\beta) - (\alpha+\beta)]}{1-\alpha} a_k \; \leq 1. \tag{4.4}$$

Hence (4.3) will be satisfied if

$$\frac{k(c+k)}{c+1} \, |z|^{k-1} < \frac{k^n[k(1+\beta) - (\alpha+\beta)]}{1-\alpha},$$

that is, if

$$|z| < \left\{ \frac{k^{n-1}[k(1+\beta) - (\alpha+\beta)](c+1)}{(1-\alpha)(c+k)} \right\}^{\frac{1}{k-1}} \quad (k \geq 2). \tag{4.5}$$

Therefore the function $f(z)$ given by (4.1) is univalent in $|z| < R^*$. The sharpness of the result follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{k^n[k(1+\beta) - (\alpha+\beta)](c+1)} z^k \quad (k \geq 2). \tag{4.6}$$

□

5. MODIFIED HADAMARD PRODUCTS

Let the functions $f_\nu(z)$ $(\nu = 1, 2)$ be defined by

$$f_\nu(z) = z - \sum_{k=2}^{\infty} a_{k,\nu} z^k \quad (a_{k,\nu} \geq 0,\ \nu = 1, 2). \tag{5.1}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k. \tag{5.2}$$

THEOREM 8. *Let each of the functions $f_\nu(z)$ $(\nu = 1, 2)$ defined by (5.1) be in the class $T(n, \alpha, \beta)$. Then $(f_1 * f_2)(z) \in T(n, \delta(n, \alpha, \beta), \beta)$, where*

$$\delta(n, \alpha, \beta) = 1 - \frac{(1+\beta)(1-\alpha)^2}{2^n(2-\alpha+\beta)^2 - (1-\alpha)^2}. \tag{5.3}$$

The result is sharp.

Proof. Employing the techniques used by Schild and Silverman in [15], we need to find the largest $\delta = \delta(n, \alpha, \beta)$ such that

$$\sum_{k=2}^{\infty} \frac{k^n[k(1+\beta) - (\delta+\beta)]}{1-\delta} a_{k,1} a_{k,2} \leq 1. \tag{5.4}$$

Since

$$\sum_{k=2}^{\infty} \frac{k^n[k(1+\beta) - (\alpha+\beta)]}{1-\alpha} a_{k,1} \leq 1 \tag{5.5}$$

and

$$\sum_{k=2}^{\infty} \frac{k^n[k(1+\beta) - (\alpha+\beta)]}{1-\alpha} a_{k,2} \leq 1, \tag{5.6}$$

the Cauchy-Schwarz inequality yields

$$\sum_{k=2}^{\infty} \frac{k^n[k(1+\beta) - (\alpha+\beta)]}{1-\alpha} \sqrt{a_{k,1} a_{k,2}} \leq 1. \tag{5.7}$$

Thus it is sufficient to show that

$$\frac{k^n[k(1+\beta) - (\delta+\beta)]}{1-\delta} a_{k,1} a_{k,2} \leq \frac{k^n[k(1+\beta) - (\alpha+\beta)]}{1-\alpha} \sqrt{a_{k,1} a_{k,2}} \tag{5.8}$$

for $k \geq 2$, that is, that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{[k(1+\beta) - (\alpha+\beta)](1-\delta)}{[k(1+\beta) - (\delta+\beta)](1-\alpha)} \quad (k \geq 2). \tag{5.9}$$

Note that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{(1-\alpha)}{k^n[k(1+\beta) - (\alpha+\beta)]} \quad (k \geq 2). \tag{5.10}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{k^n[k(1+\beta)-(\alpha+\beta)]} \le \frac{[k(1+\beta)-(\alpha+\beta)](1-\delta)}{[k(1+\beta)-(\delta+\beta)](1-\alpha)} \quad (k \ge 2), \tag{5.11}$$

or, equivalently, that

$$\delta \le 1 - \frac{(k-1)(1+\beta)(1-\alpha)^2}{k^n[k(1+\beta)-(\alpha+\beta)]^2-(1-\alpha)^2} \quad (k \ge 2). \tag{5.12}$$

Since

$$\Phi(k) = 1 - \frac{(k-1)(1+\beta)(1-\alpha)^2}{k^n[k(1+\beta)-(\alpha+\beta)]^2-(1-\alpha)^2} \tag{5.13}$$

is an increasing function of $k\,(k \ge 2)$, letting $k=2$ in (5.13), we obtain

$$\delta \le \Phi(2) = 1 - \frac{(1+\beta)(1-\alpha)^2}{2^n(2-\alpha+\beta)^2-(1-\alpha)^2}, \tag{5.14}$$

which proves the main assertion of Theorem 8.

Finally, by taking the functions $f_\nu(z)$ $(\nu = 1,2)$ given by

$$f_\nu(z) = z - \frac{1-\alpha}{2^n(2-\alpha+\beta)}z^2 \quad (\nu = 1,2), \tag{5.15}$$

we can see that the result is sharp. □

Proceeding as in the proof of Theorem 8, we get

THEOREM 9. *Let the functions* $f_1(z)$ *and* $f_2(z)$ *defined by* (5.1) *be in the classes* $T(n,\alpha,\beta)$ *and* $T(n,\gamma,\beta)$*, respectively. Then*

$$(f_1 * f_2)(z) \in T(n, \xi(n,\alpha,\gamma,\beta),\beta),$$

where

$$\xi(n,\alpha,\gamma,\beta) = 1 - \frac{(1+\beta)(1-\alpha)(1-\gamma)}{2^n(2-\alpha+\beta)(2-\gamma+\beta)-(1-\alpha)(1-\gamma)}. \tag{5.16}$$

The result is the best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{2^n(2-\alpha+\beta)}z^2, \quad f_2(z) = z - \frac{1-\gamma}{2^n(2-\gamma+\beta)}z^2\,. \tag{5.17}$$

THEOREM 10. *Let the functions* $f_\nu(z)$ $(\nu = 1,2)$ *defined by* (5.1) *be in the class* $T(n,\alpha,\beta)$*. Then the function*

$$h(z) = z - \sum_{k=2}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k \tag{5.19}$$

belongs to the class $T(n,\tau(n,\alpha,\beta),\beta)$*, where*

$$\tau(n,\alpha,\beta) = 1 - \frac{(1+\beta)(1-\alpha)^2}{2^{n-1}(2-\alpha+\beta)^2-(1-\alpha)^2}. \tag{5.20}$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1,2)$ *defined by* (5.15).

6. PROPERTIES ASSOCIATED WITH GENERALIZED FRACTIONAL CALCULUS OPERATORS

In terms of the Gauss hypergeometric function

$$ {}_2F_1(\delta,\mu;\nu;z)=\sum_{k=0}^{\infty}\frac{(\delta)_k(\mu)_k}{(\nu)_k}\frac{z^k}{k!} \tag{6.1}$$

for $z\in U$, $\delta,\mu,\nu\in C$, $\nu\neq 0,-1,-2,\dots$, where $(\lambda)_k$ denotes the Pochhammer symbol defined, in terms of the Gamma functions, by

$$(\lambda)_k=\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}=\begin{cases}1 & (k=0)\\ \lambda(\lambda+1)\dots(\lambda+k-1) & (k\in\mathbb{N}).\end{cases}$$

The generalized fractional calculus operators $I_{0,z}^{\mu,\nu,\eta}$ and $J_{0,z}^{\mu,\nu,\eta}$ are defined below (cf., e.g., [8] and [18]).

Definition 3. (The generalized fractional integral operators.) The generalized fractional integral of order μ is defined, for a function $f(z)$, by

$$I_{0,z}^{\mu,\nu,\eta}f(z)=\frac{z^{-\mu-\nu}}{\Gamma(\mu)}\int_0^z(z-\zeta)^{\mu-1}\,{}_2F_1\left(\mu+\nu;-\eta;\mu;1-\frac{\zeta}{z}\right)f(\zeta)\mathrm{d}\zeta \tag{6.2}$$

for $\mu>0$, $\epsilon>\max\{0,\nu-\eta\}-1$, where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$, provided further that

$$f(z)=O(|z|^{\epsilon})\ (z\to 0). \tag{6.3}$$

Definition 4. (The generalized fractional derivative operators.) The generalized fractional derivative of order μ is defined, for a function $f(z)$, by

$$J_{0,z}^{\mu,\nu,\eta}f(z)=\begin{cases}\frac{1}{\Gamma(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^{\mu-\nu}\int_0^z(z-\zeta)^{-\mu}\,{}_2F_1(\nu-\mu,1-\eta;1-\mu;\right.\\ \left.1-\frac{\zeta}{z})f(\zeta)\mathrm{d}\zeta\right\}\ (0\le\mu<1)\ ,\\ \frac{\mathrm{d}^n}{\mathrm{d}z^n}J_{0,z}^{\mu-n,\nu,\eta}f(z)\ (n\le\mu<n+1,\ n\in N)\end{cases} \tag{6.4}$$

for $\epsilon>\max\{0,\nu-\eta\}-1$, where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed, as in Definition 3, and ϵ is given by (6.3).

It follows from Definition 3 and Definition 4 that

$$I_{0,z}^{\mu,-\mu,\eta}f(z)=D_z^{-\mu}f(z)\ (\mu>0), \tag{6.5}$$

$$J_{0,z}^{\mu,\mu,\eta}f(z)=D_z^{\mu}f(z)\ (0\le\mu<1), \tag{6.6}$$

where $D_z^\mu(\mu \in R)$ is the fractional operator considered by Owa in [7] and (subsequently) by Owa and Srivastava in [9] and Srivastava and Owa in [17]. Furthermore, in terms of the Gamma function, Definitions 3 and 4 readily yield the following result.

LEMMA 2. ([18]) *The generalized fractional integral and the generalized fractional derivative of a power function are given by*

$$I_{0,z}^{\mu,\nu,\eta} z^\rho = \frac{\Gamma(\rho+1)\Gamma(\rho-\nu+\eta+1)}{\Gamma(\rho-\nu+1)\Gamma(\rho+\mu+\eta+1)} z^{\rho-\nu} \tag{6.7}$$

for $\mu > 0$, $\rho > \max\{0, \nu-\eta\} - 1$, *and*

$$J_{0,z}^{\mu,\nu,\eta} z^\rho = \frac{\Gamma(\rho+1)\Gamma(\rho-\nu+\eta+1)}{\Gamma(\rho-\nu+1)\Gamma(\rho-\mu+\eta+1)} z^{\rho-\nu} \tag{6.8}$$

for $0 \le \mu < 1$, $\rho > \max\{0, \nu-\eta\} - 1$.

THEOREM 11. *Let* $f(z)$ *defined by* (1.18) *be in the class* $T(n, \alpha, \beta)$. *Then*

$$\begin{aligned} &\frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2+\mu+\eta)} |z|^{1-\nu} \left\{1 - \tfrac{(1-\alpha)(2-\nu+\eta)}{2^{n-1}(2-\alpha+\beta)(2-\nu)(2+\mu+\eta)} |z|\right\} \\ &\le \left| I_{0,z}^{\mu,\nu,\eta} f(z) \right| \\ &\le \frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2+\mu+\eta)} |z|^{1-\nu} \left\{1 + \tfrac{(1-\alpha)(2-\nu+\eta)}{2^{n-1}(2-\alpha+\beta)(2-\nu)(2+\mu+\eta)} |z|\right\} \end{aligned} \tag{6.9}$$

for $z \in U_0$, $\mu > 0$, $\max\{\nu, \nu-\eta, -\mu-\eta\} < 2$, $\nu(\mu+\eta) \le 3\mu$, *and*

$$\begin{aligned} &\frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2-\mu+\eta)} |z|^{1-\nu} \left\{1 - \tfrac{(1-\alpha)(2-\nu+\eta)}{2^{n-1}(2-\alpha+\beta)(2-\nu)(2-\mu+\eta)} |z|\right\} \\ &\le \left| J_{0,z}^{\mu,\nu,\eta} f(z) \right| \\ &\le \frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2-\mu+\eta)} |z|^{1-\nu} \left\{1 + \tfrac{(1-\alpha)(2-\nu+\eta)}{2^{n-1}(2-\alpha+\beta)(2-\nu)(2-\mu+\eta)} |z|\right\} \end{aligned} \tag{6.10}$$

for $z \in U_0$, $0 \le \mu < 1$, $\max\{\nu, \nu-\eta, \mu-\eta\} < 2$, $\nu(\mu-\eta) \ge 3\mu$, *where*

$$U_0 = \begin{cases} U & (\nu \le 1) \\ U \backslash \{0\} & (\nu > 1) \end{cases} . \tag{6.11}$$

Each of these results is sharp for the function $f(z)$ *defined by* (2.2).

Proof. First of all, since the function $f(z)$ defined by (1.18) is in the class $T(n, \alpha, \beta)$, we can apply Lemma 1 to deduce that

$$\sum_{k=2}^{\infty} a_k \le \frac{1-\alpha}{2^n(2-\alpha+\beta)} . \tag{6.12}$$

Next, making use of the assertion (6.7) of Lemma 2, we find from (1.18) that

$$F(z) = \frac{\Gamma(2-\nu)\Gamma(2+\mu+\eta)}{\Gamma(2-\nu+\eta)} z^\nu \, I_{0,z}^{\mu,\nu,\eta} f(z) = z - \sum_{k=2}^{\infty} \Phi(k) a_k z^k, \tag{6.13}$$

where, for convenience,

$$\Phi(k)=\frac{(1)_k(2-\nu+\eta)_{k-1}}{(2-\nu)_{k-1}(2+\mu+\eta)_{k-1}}\quad (k\in\mathbb{N}\backslash\{1\}). \tag{6.14}$$

The function $\Phi(k)$ defined by (6.14) is nonincreasing under the parametric constraints stated already with (6.9), and we thus have

$$0<\Phi(k)\le\Phi(2)=\frac{2(2-\nu+\eta)}{(2-\nu)(2+\mu+\eta)}\quad (k\in\mathbb{N}\backslash\{1\}). \tag{6.15}$$

Now the assertion (6.9) of Theorem 11 follows from (6.12) and (6.15).

The inequalities (6.10) can be proved similarly, observing that from (6.8) we get

$$G(z)=\frac{\Gamma(2-\nu)\Gamma(2-\mu+\eta)}{\Gamma(2-\nu+\eta)}z^{\nu}\, J_{0,z}^{\mu,\nu,\eta}f(z)=z-\sum_{k=2}^{\infty}\Psi(k)a_k z^k, \tag{6.16}$$

where

$$\begin{aligned}0<\Psi(k)&=\frac{(1)_k(2-\nu+\eta)_{k-1}}{(2-\nu)_{k-1}(2-\mu+\eta)_{k-1}}\\ &\le\Psi(2)=\frac{2(2-\nu+\eta)}{(2-\nu)(2-\mu+\eta)}\quad (k\in\mathbb{N}\backslash\{1\}),\end{aligned} \tag{6.17}$$

under the parametric constraints stated already with (6.10).

Finally, by observing that the equalities in each of the assertions (6.9) and (6.10) are attained by the function $f(z)$ given by (2.2), we complete the proof of Theorem 11. □

In view of the relationships (6.5) and (6.6), by setting $\nu=-\mu$ and $\nu=\mu$ in our assertions (6.9) and (6.10), respectively, we obtain the following result.

COROLLARY 4. *Let $f(z)$ defined by* (1.18) *be in the class $T(n,\alpha,\beta)$. Then*

$$\begin{aligned}&\frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1-\frac{1-\alpha}{2^{n-1}(2-\alpha+\beta)(2+\mu)}\,|z|\right\}\le\left|D_z^{-\mu}f(z)\right|\\ &\le\frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1+\frac{1-\alpha}{2^{n-1}(2-\alpha+\beta)(2+\mu)}\,|z|\right\}(z\in U;\mu>0)\,.\end{aligned} \tag{6.18}$$

The result is sharp for the function $f(z)$ given by (2.2).

REMARK 2. Note that the result obtained by Rosy and Murugusundaramoorthy in [13, Corollary 2] is not correct. The correct result is given by (6.18).

COROLLARY 5. *Let $f(z)$ defined by* (1.18) *be in the class $T(n,\alpha,\beta)$. Then*

$$\begin{aligned}&\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1-\frac{1-\alpha}{2^{n-1}(2-\alpha+\beta)(2-\mu)}\,|z|\right\}\le|D_z^{\mu}f(z)|\\ &\le\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1+\frac{1-\alpha}{2^{n-1}(2-\alpha+\beta)(2-\mu)}\,|z|\right\}(z\in U;0\le\mu<1)\,.\end{aligned} \tag{6.19}$$

The result is sharp for the function $f(z)$ given by (2.2).

REMARK 3. Note that the result obtained by Rosy and Murugusundaramoorthy in [13, Corollary 3] is not correct. The correct result is given by (6.19).

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Received November 7, 2007
Accepted April 17, 2009

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