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TRANSPORT OF STRUCTURE VIA MÖBIUS FUNCTIONS

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Abstract. We show that certain group or field structures appearing in high school Algebra textbooks arise by transport of known structures via Möbius functions.

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1. INTRODUCTION

A widely encountered problem in the Algebra section of late high school can be identified as:

Given operations on a set, show that the structure is isomorphic to a known structure, knowing that the isomorphism has a certain form.

We will consider here the following two types of problems:

Type 1. On the set $\mathbb{R}$ of real numbers consider the commutative operations:

$$x \perp y = a_1x + a_1y + b_1$$
$$x * y = a_2xy + b_2x + b_2y + c_2$$

Find the real parameters a_i, b_i, c_i such that the structure $(\mathbb{R}, \perp, *)$ is a field. Find also the parameters α and β such that a function of the form

$$f : (\mathbb{R}, \perp, *) \to (\mathbb{R}, +, \cdot), \qquad f(x) = \alpha x + \beta$$

is a field isomorphism. Here we assume that $a_1, a_2 \neq 0$.

Type 2. On the interval $G \subset \mathbb{R}$ consider the commutative operation:

$$x * y = \frac{\alpha_1xy + \beta_1x + \beta_1y + \gamma_1}{\alpha_2xy + \beta_2x + \beta_2y + \gamma_2}$$

Find the parameters such that the structure $(G, *)$ is an abelian group. Find also the parameters a, b, c, d such that a function of the form

$$f : (G, *) \to (\mathbb{R}^*_+, \cdot), \qquad f(x) = \frac{ax + b}{cx + d}$$

is a group isomorphism. Here we assume that $c \neq 0$.

A read through a set of twelfth grade textbooks yielded the following exercises, among many similar ones.

EXAMPLE 1.1. Let $a, b, c \in \mathbb{R}$. On $\mathbb{R}$ we define the following operations

$$x \perp y = ax + by - 2,$$
$$x * y = xy - 2x - 2y + c.$$

(1) Find a, b, c such that $(\mathbb{R}, \perp, *)$ is a field.
(2) Find $\alpha, \beta \in \mathbb{R}$ such that the function $f : \mathbb{R} \to \mathbb{R}, f(x) = \alpha x + \beta$ is a field isomorphism between $(\mathbb{R}, +, \cdot)$ and $(\mathbb{R}, \perp, *)$.

[2, Exercise 7, p. 67]

EXAMPLE 1.2. On the set $\mathbb{Z}$ of integers consider the operations

$$x \perp y = x + y - 2$$
$$x * y = xy - 2(x + y) + 6$$

Prove that a function of the form $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = ax + b$ is a ring isomorphism between $(\mathbb{Z}, \perp, *)$ and $(\mathbb{Z}, +, \cdot)$.

[2, Test 1, Exercise 1, p. 68]

EXAMPLE 1.3. Prove that the set $G = (-1, 1)$ together with the operation $x * y = \frac{x+y}{1+xy}$ forms a group . Show also that $(G, *)$ and $(\mathbb{R}_+^*, \cdot)$ are isomorphic.

[3, Exercise 2, p. 42]

EXAMPLE 1.4. On the interval $G = (0, 2)$ define the operation $x \circ y = \frac{xy}{xy-x-y+2}$.

(1) Show that $(G, \circ)$ is an abelian group.
(2) Show that $f : (0, 2) \to (0, \infty), f(x) = \frac{2-x}{x}$ is a group isomorphism.

[3, Exercise 8, p. 44]

There are definitely many other examples and variants, but these are already enough as a motivation for the present article. We are going to discuss these problems by systematically using transport of structure. Note that this notion is presented in some of the textbooks, see for instance [1, Section 3.3] and [4, p. 48-49]. We show here that the above examples and similar ones are obtained by transport of structure via Möbius functions.

2. TYPE 1: FIELDS ISOMORPHIC TO THE FIELD OF REAL NUMBERS

As in the Introduction, on $\mathbb{R}$ we consider the operations:

$$x \perp y = a_1 x + a_1 y + b_1$$
$$x * y = a_2 xy + b_2 x + b_2 y + c_2$$

and the bijective function

$$f : (\mathbb{R}, \perp, *) \to (\mathbb{R}, +, \cdot), \qquad f(x) = \alpha x + \beta,$$

so $\alpha \neq 0$. The parameters $a_1, b_1, a_2, b_2, c_2 \in \mathbb{R}$ will be determined from the assumption that f is a field isomorphism, that is, for all $x, y \in \mathbb{R}$ we have:

$$f(x \perp y) = f(x) + f(y)$$
$$f(x * y) = f(x) \cdot f(y).$$

We have $f^{-1}(x) = \frac{1}{\alpha}x - \frac{\beta}{\alpha}$, hence

$$x \perp y = f^{-1}(f(x) + f(y))$$
$$x * y = f^{-1}(f(x) \cdot f(y)).$$

Easy computations will yield

$$x \perp y = f^{-1}(\alpha(x+y) + 2\beta) = \frac{1}{\alpha}(\alpha(x+y) + 2\beta) - \frac{\beta}{\alpha} = x + y + \frac{\beta}{\alpha}$$

and

$$\begin{aligned} x * y &= f^{-1}((\alpha x + \beta)(\alpha y + \beta)) \\ &= f^{-1}(\alpha^2 xy + \alpha\beta(x+y) + \beta^2) \\ &= \frac{1}{\alpha}(\alpha^2 xy + \alpha\beta x + \alpha\beta y + \beta^2) - \frac{\beta}{\alpha} \\ &= \alpha xy + \beta x + \beta y + \frac{\beta^2 - \beta}{\alpha}. \end{aligned}$$

Therefore, it follows that if

$$x \perp y = x + y + \frac{\beta}{\alpha} \tag{1}$$

$$x * y = \alpha xy + \beta x + \beta y + \frac{\beta^2 - \beta}{\alpha}, \tag{2}$$

then the function

$$f : (\mathbb{R}, \perp, *) \to (\mathbb{R}, +, \cdot), \qquad f(x) = \alpha x + \beta$$

is a field isomorphism.

Remark 1. The zero element of the field $(\mathbb{R}, \perp, *)$ is $f^{-1}(0) = -\frac{\beta}{\alpha}$ and its unit element is $f^{-1}(1) = \frac{1-\beta}{\alpha}$.

Actually, we have the following converse result, which is a kind of uniqueness property.

Theorem 1. *Assume that* $(\mathbb{R}, \perp, *)$ *is a field. Then* $(\mathbb{R}, \perp, *)$ *is isomorphic to* $(\mathbb{R}, +, \cdot)$.

Proof. We start by considering d to be the neutral element of $(\mathbb{R}, \perp)$, that is, $x \perp d = x$ for all $x \in \mathbb{R}$, which explicitly translates to

$$\begin{aligned} a_1 x + a_1 d + b_1 = x &\Leftrightarrow x(a_1 - 1) + a_1 d + b_1 = 0 \\ &\Leftrightarrow a_1 = 1 \text{ and } d = -b_1. \end{aligned}$$

Therefore, $\perp$ has the form $x \perp y = x + y + b_1$, with $d = -b_1$ being its neutral element. Considering x' to be the symmetric of x with regard to $\perp$, short computations will yield that $x' = -2b_1 - x$.

At this point, we already have that $(\mathbb{R}, \perp) \cong (\mathbb{R}, +)$, the isomorphism being $f : \mathbb{R} \to \mathbb{R}, f(x) = x - b_1$.

Next, consider e being the neutral element with respect to $*$. It follows that for all $x \in \mathbb{R}$

$$a_2xe + b_2x + b_2e + c_2 = x \Leftrightarrow x(a_2e + b_2 - 1) + b_2e + c_2 = 0$$
$$\Leftrightarrow e = \frac{1 - b_2}{a_2} = -\frac{c_2}{b_2}$$

Note that $a_2 \neq 0$. Computing this will yield that

$$e = \frac{b_2^2 - b_2}{a_2}.$$

It follows that

$$x * y = a_2xy + b_2(x + y) + \frac{b_2^2 - b_2}{a_2}.$$

For the sake of ease, rewrite the expression of $\perp$ as

$$x \perp y = x + y + d,$$

having $-d$ as a neutral (zero) element. Keeping in mind that $(\mathbb{R}, \perp, *)$ is a field, it is known that $x * (-d) = (-d)$ for all $x \in \mathbb{R}$. This directly translates to

$$-a_2dx + b_2x - b_2d + \frac{b_2^2 - b_2}{a_2} + d = 0,$$

that is,

$$x(b_2 - a_2d) + \frac{b_2^2 - b_2 - a_2b_2d + a_2d}{a_2} = 0.$$

Since this happens for all $x \in \mathbb{R}$, it follows that

$$b_2 = a_2d \quad \text{and} \quad a_2^2d^2 - a_2d - a_2^2d^2 + ad = 0.$$

Finally, dropping the indices for ease of writing, this yields the final forms of the given operations:

$$x \perp y = x + y + d \tag{3}$$

$$x * y = axy + ad(x + y) + \frac{a^2d^2 - ad}{a}. \tag{4}$$

with the zero element being $-d$, and the unit element being $\frac{1-ad}{a}$.

Comparing (3) and (4) with (1) and (2), we obtain that the map

$$f : (\mathbb{R}, \perp, *) \to (\mathbb{R}, +, \cdot), \qquad f(x) = \alpha x + \beta,$$

is a field isomorphism for $\alpha = a$ and $\beta = ad$, that is, $f(x) = ax + ad$. □

Remark 2. It is clear from the above calculations that instead of the field $(\mathbb{R}, +, \cdot)$ of real numbers, we could have started with any field $(F, +, \cdot)$ and obtain the same results.

Actually, we may even start with an integral domain $(A, +, \cdot)$, but in that case we must have that α is an invertible element of A.

3. TYPE 2: GROUPS ISOMORPHIC TO $(\mathbb{R}_+^*, \cdot)$

Ias in the Introduction, consider an interval $G \subset \mathbb{R}$ and the operation

$$x * y = \frac{\alpha_1 xy + \beta_1 x + \beta_1 y + \gamma_1}{\alpha_2 xy + \beta_2 x + \beta_2 y + \gamma_2}.$$

As above, the parameters will be determined from hypothesis that the map

$$f : (G, *) \to (\mathbb{R}_+^*, \cdot), \qquad f(x) = \frac{ax+b}{cx+d}$$

is a continuous group isomorphism, where we assume that $c \neq 0$.

We first recall some well-known facts on the group of Möbius transformations. Consider the set of functions

$$\mathcal{M} = \{f : \overline{\mathbb{R}} \to \overline{\mathbb{R}} \mid f(x) = \frac{ax+b}{cx+d},\ ad - bc \neq 0\},$$

with forms a group together with composition of functions. Consider also the group morphism

$$\phi : (\mathrm{GL}_2(\mathbb{R}), \cdot) \to (\mathcal{M}, \circ), \qquad \phi(A) = \frac{ax+b}{cx+d}.$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The kernel of this morphism is, by definition,

$$\mathrm{Ker}(\phi) = \{A \mid \phi(A) = \mathbf{1}_{\bar{\mathbb{R}}}\}.$$

This above condition translates to

$$ax + b = cx^2 + dx, \text{ for all } x \in \mathbb{R},$$

from which we get $c = b = 0$, and $a = d \in \mathbb{R}^*$. It follows that

$$\mathrm{Ker}(\phi) = Z(\mathrm{GL}_2(\mathbb{R})) = \{aI_2 \mid a \in \mathbb{R}^*\}.$$

Therefore, by the First Isomorphism Theorem, ϕ induces the isomorphism

$$(\mathrm{PGL}_2(\mathbb{R}), \cdot) \simeq (\mathcal{M}, \circ).$$

We have that $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. It follows that if $\phi(A) = f$, then

$$f^{-1}(x) = \frac{dx - b}{-cx + a}.$$

Remark 3. Given the above isomorphism between the Möbius group and the projective linear group of degree 2, we see that it is no loss if we assume that $c = 1$.

Returning to our problem, the following step is to explicitly determine the operation $*$ from the group morphism condition $f(x*y)=f(x)f(y)$. On one hand, directly computing the previous expression will yield

$$\frac{ax*y+b}{cx*y+d}=\frac{ax+b}{cx+d}\cdot\frac{ay+b}{cy+d}=\frac{a^2xy+abx+aby+b^2}{c^2xy+cdx+cdy+d^2} \tag{5}$$

On the other hand, we have

$$x*y=f^{-1}(f(x)f(y)).$$

Applying this to the equation (5), it follows that

$$\begin{aligned}x*y&=\frac{d\cdot\frac{a^2xy+abx+aby+b^2}{c^2xy+cdx+cdy+d^2}-b}{-c\cdot\frac{a^2xy+abx+aby+b^2}{c^2xy+cdx+cdy+d^2}+a}\\&=\frac{(da^2-bc^2)xy+(abd-bcd)x+(abd-bcd)y+db^2-bd^2}{(ac^2-ca^2)xy+(acd-abc)x+(acd-abc)y+da^2-cb^2},\end{aligned}$$

where recall that we may assume that $c=1$.

Now that the operation has been determined, the next step is determining the interval G. The condition of f being continuous yields

$$f(x)=0\Leftrightarrow x=-\frac{b}{a}$$

and

$$f(x)=\infty\Leftrightarrow x=-\frac{d}{c}.$$

Therefore, depending on whichever is the greatest value, G can either be of form $(-\frac{b}{a},-\frac{c}{d})$ or of the form $(-\frac{c}{d},-\frac{b}{a})$. Since $c\neq 0$, it follows that G is an interval bounded at least at one end.

Finally, the equation $f(x)=y\in(0,\infty)$ will yield that $x=\frac{dy-b}{-cy+a}$. This solution does not exist for $y=\frac{a}{c}$. Therefore, $\frac{a}{c}\notin(0,\infty)$. But since we may assume that $c=1$, it follows that we must take $a\leq 0$.

As a conclusion to the current section, if

$$x*y=\frac{(da^2-b)xy+(abd-bd)x+(abd-bd)y+db^2-bd^2}{(a-a^2)xy+(ad-ab)x+(ad-ab)y+da^2-b^2}$$

is an operation defined on the interval $G=(-\frac{b}{a},-\frac{d}{c})$ or $G=(-\frac{d}{c},-\frac{b}{a})$, then the map

$$f:(G,*)\to(0,\infty),\qquad f(x)=\frac{ax+b}{x+d}$$

is a group isomorphism, where $a\leq 0$.

Remark 4. The neutral element of $(G,*)$ is $f^{-1}(1)=\frac{d-b}{a-c}$.

4. APPLICATIONS

As applications, we will show that each of the examples presented in the introduction fits into one on the two types.

Example 1.1 clearly fits the first pattern with $a_1 = a = b$, $b_1 = -2$, $a_2 = 1$, $b_2 = -2$, $c_2 = c$. Applying the reasoning in the first section will yield the isomorphism $f(x) = x - 2$.

Same goes for Example 1.2, having $a_1 = 1$, $b_1 = -2$, $a_2 = 1$, $b_2 = -2$, $c_2 = 6$. Analogously, the isomorphism in this case is $f(x) = x - 2$.

As far as Example 1.3 goes, it fits the second pattern, having $\alpha_1 = 0$, $\beta_1 = 1$, $\gamma_1 = 0$, $\alpha_2 = 1$, $\beta_2 = 0$, $\gamma_2 = 1$.

Similarly for Example 1.4, we have $\alpha_1 = 1$, $\beta_1 = \gamma_1 = 0$, $\alpha_2 = 1$, $\beta_2 = -1$, $\gamma_2 = 2$. Their respective isomorphisms can also be determined using the above algorithms.

Finally, we leave to the reader the following question inspired by Theorem 1. With the notations used in the previous section, assume that $(G, *)$ is an abelian group. Does it follow that $(G, *)$ is isomorphic to $(\mathbb{R}_+^*, \cdot)$?

REFERENCES

[1] Andrica, D. ș. a. *Matematică. Manual pentru clasa a XII-a: M1*, Editura Plus, București, 2002.

[2] Ion, D. I. ș. a. *Matematică. Manual pentru clasa a XII-a: M1*, Editura SIGMA, București, 2007.

[3] Radu, E., Șontea, O., *Matematică. Manual pentru clasa a XII-a: M1*, Editura ALL, Bucureşti 2012.

[4] Țena, M. ș. a. *Matematică. Manual pentru clasa a XII-a: M1*, Editura ART, București, 2010.

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