**DIDACTICA MATHEMATICA, Vol. 37 (2019) , pp. 17–28**

# TOPOLOGY ON $\mathbb{R}$ AND $\overline{\mathbb{R}}$. CHALLENGES IN TEACHING IT TO FIRST YEAR STUDENTS.

Anca GRAD

**Abstract.** It is a common practice for the high-school students to compute derivatives of certain functions by applying formulas, ignoring the depth of the problem concerning the actual domain on which such derivatives should be studied. Understanding thoroughly the concept of the set of accumulation points holds the key in correctly studying limits of functions, derivatives and obviously, asymptotes. In this article we give the definition and characterization of the set of accumulation points, accompanied by some tricky examples which challenge the understanding of the students, as observed in the years of practice with Mathematics and Mathematics and Computer Science students at the lecture and seminar of Calculus 1.

**MSC 2000.** 54H25.

**Key words.** topology, convergence, the set of the accumulation points

## 1. INTRODUCTION

Starting with the university year 2017-2018, the study programme Mathematics for Computer Science, in English, has been offered to students at the Faculty of Mathematics and Computer Science, at Babeş-Bolyai University of Cluj-Napoca. The first year students display good knowledge in English, however the adjustment to specific Calculus terminology slows down a bit the paste of the first lectures. In addition, the first topic under study, i.e. topology on $\mathbb{R}$, due to its high degree of formalization, despite being quite basic, raises problems in understanding.

## 2. PRLIMINARIES

The notions and approaches presented within this article are considered within the framework of the real topological vector space $\mathbb{R}$, over the field of scalars $\mathbb{R}$. Due to the fact that such algebraic notions are presented to first years students within specific Algebra lectures in the middle of the first term, in Calculus, at the first lecture, $\mathbb{R}$ is presented as being seen as a commutative field with its two operations

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

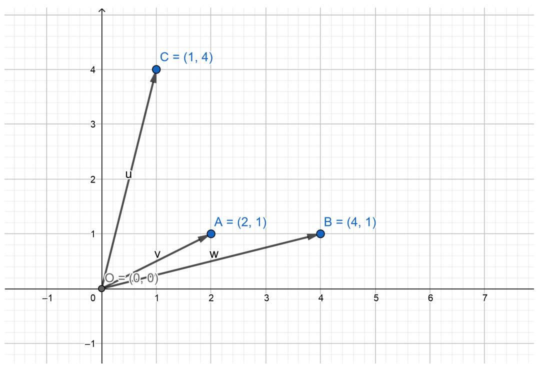
and

$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Basically speaking, all the analysis of such real numbers can be imagined graphically as happening on a never-ending line, whose center is 0. This is why, the students must be encouraged to use this representation as often as possible while searching for possible solutions of their problems. The notions and results are presented such that, whenever they encounter in the future a similar notion, stated in a different space, to be able to spot the similarities and distinctions. Thus, sometimes, they are left with the impression that the teachers stresses to much on certain aspects that are quite obvious. In such cases, it is desirable for the teacher to provide examples in higher order spaces (usually $\mathbb{R}^2$ or $\mathbb{R}^3$) where the property is easier to envision, or does not even hold. Such an example is the situation when we must **carefully** emphasize the fact that $\mathbb{R}$ is **totally ordered**, by the order relation $\leq$ i.e.

$$\forall x, y \in \mathbb{R}, \quad \text{it holds either} \quad x \leq y \quad \text{or} \quad y \leq x.$$

Without an example in $\mathbb{R}^2$, this notion seems pointless to students. I often use the picture:




The obvious question is how to order the points $A, B$ and $C$. In most of the cases especially if the audience is numerous, there is a voice in the crowd saying that the order depends on the criteria that we choose (the distance to one of the axis, or the distance to the origin). But by using such criteria, we get distinct points which are at the same time equal. (For example, considering the length to the origin, the distinct points $B$ and $C$ are equal, whereas by considering the distance to the $Ox$ axis, the points $A$ and $B$ are equal). Such an abnormal behaviour does not happen in $\mathbb{R}$, when considering the order induce by the order relation $\leq$.

In contrast to algebra, where always exact solutions are searched for, analysis uses lots of approximations, and this is why we are not only interested in a certain number (natural, integer, rational or real), but also in what happens in its neighbourhood on the real axis. In many exercises, which involve natural numbers, we find it quite handy to use the so-called **Archimedes axiom**, which states that the set of natural numbers is unbounded, and is mostly used in one of the following two equivalent variants:

$$\forall x > 0, \quad \exists n_x \in \mathbb{N} \quad s.t. \quad x < n_x$$

or

$$\forall y > 0, \quad \exists n_y \in \mathbb{N} \quad s.t. \quad \frac{1}{n_y} < y.$$

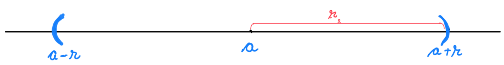
**2.1. Balls and Neighborhoods in $\mathbb{R}$.** As an advised reader well knows, topologies are build upon open sets (or equivalently on closed sets), defined through neighborhoods.

In particular the notion of a **neighborhood** is quite blurry for our first year students. By simply introducing first the definition and the main properties, the level of understanding is quite low. In time I found it more appropriate to introduce this notion in front of the blackboard, with a piece of chalk in my hands. By holding the piece of chalk with just two fingers I ask the students weather I am or not a neighborhood of the chalk. Almost every time everybody agrees that I am, and is surprised to hear that I am not. Then, I wrap my two fists around the piece of chalk, an explain to them that by doing that, I become a neighborhood. Moreover, if I add them, to me, we all are together a neighborhood of the chalk. This simple examples seals the deal with the notion, and makes is accessible to almost all the audience.

My fist, is at the same time, the best example that I can provide them with, in terms of balls. They can see clearly that it looks like one, and it contains the object. This theory works well in the third dimension, i.e. $\mathbb{R}^3$, when we consider the Euclidian norm. In order to be able to understand the concepts on $\mathbb{R}$, we have to recall the fact that the set of the real numbers is represented as a line. Thus, if we imagine a needle going through my fist, the intersection of the needle with it gives us the representation of a ball in this first dimension,

In theory, having a random real number $a \in \mathbb{R}$, and a random positive constant $r \in \mathbb{R}_+^*$, we define **the (open) ball** of center $a$ and radius $r$), the set

$$B(a,r) = (a-r, a+r) = \{x \in \mathbb{R} : |x-a| < r\}.$$




A random set $V \subseteq \mathbb{R}$ is said to be a **neighbourhood** of $a \in \mathbb{R}$, if there exists an $r > 0$ such that

$$B(a,r) \subseteq V.$$

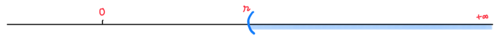
The set of all neighbourhoods associated to the point $a$ is denoted by $\mathcal{V}(a)$.

Example 2.1. *Consider the point* $2 \in \mathbb{R}$. *The interval* $(1,3) \in \mathcal{V}(2)$ *and everything reunited with it is going to remain a neighbourhood of* $2$. *Hence,*

$$(1,3] \cup \{4\}, \quad (1,3) \cup \mathbb{N}, \quad (1,3) \cup \mathbb{Q}, \quad \mathbb{R} \in \mathcal{V}(a).$$

**2.2. Balls and Neighborhoods in $\overline{\mathbb{R}}$.** In terms of real numbers, the theory concerning neighbourhoods remains almost the same, the single difference is that the set $V$ is take from $\overline{\mathbb{R}}$ instead of $\mathbb{R}$. The catch now is that we have to define neighbourhoods for both $\infty$ and $-\infty$, and in order to provide them somehow similarly to those of real numbers, we need definitions for balls centered at those elements. Thus, considering a real positive number $r > 0$, **the open ball** of center $\infty$ and radius $r$ is the set

$$B(\infty, r) = (r, \infty] = \{x \in \mathbb{R} : x > r\},$$




and **the open ball** of center $-\infty$ and radius $r$ is the set

$$B(-\infty, r) = [-\infty, -r) = \{x \in \mathbb{R} : x < -r\}.$$


Having all these three types of balls defined, one can give a unified definition for a neighbourhood in the extended real space. Thus, a random set $V \subseteq \overline{\mathbb{R}}$ is said to be a **neighbourhood** of $a \in \overline{\mathbb{R}}$, if there exists an $r > 0$ such that

$$B(a, r) \subseteq V.$$

EXAMPLE 2.2. a) *The case when $a \in \mathbb{R}$. In this case, all the examples given in Example 2.1 remain valid. Moreover, we may add to them, $\infty$ or $-\infty$.*

b) *The case when $a = \infty$. Then $(1, \infty]$ is the ball centered at $\infty$ of radius 1, thus each set which is formed by its reunion to something else is going to remain a neighbourhood of $\infty$. Hence*

$$(1, \infty], \quad (1, \infty] \cup \mathbb{Z}, \quad (1, \infty] \cup \mathbb{Q}, \quad (1, \infty] \cup \overline{\mathbb{R}} \quad \in \mathcal{V}(\infty).$$

c) *The case when $a = -\infty$. Then$[-\infty, -1)$ is the ball centered at $-\infty$ of radius 1, thus each set which is formed by its reunion to something else is going to remain a neighbourhood of $\infty$. Hence*

$$[-\infty, -1) \quad [-\infty, -1) \cup \mathbb{Z}, \quad [-\infty, -1) \cup \mathbb{Q}, \quad \overline{\mathbb{R}} \quad \in \mathcal{V}(-\infty).$$

Deciding which topology to use, is mostly dictated by the problem under consideration. In my experience, students find it a lot more challenging to work with neighbourhoods of either $\infty$ or $-\infty$, that is why, I usually use the first two seminars on this topic to mainly speak just about properties of the topology on $\mathbb{R}$.

## 3. BASIC PROPERTIES OF NEIGHBOURHOODS IN $\mathbb{R}$

In the following we underline the basic properties of neighbours in $\mathbb{R}$.

Lemma 3.1. *Let $a \in \mathbb{R}$, and consider $V \in \mathcal{V}(a)$, a neighbourhood of $a$. Then*

$$a \in V.$$

*Proof.* From the definition of a neighbourhood, for $V \in \mathcal{V}(a)$, there exists $r_V > 0$ such that

$$B(a, r_v) \subseteq V.$$

Since $a \in B(a, r_V)$, it follows immediately that $a \in V$. □

Remark 3.1. *The converse statement in Lemma 3.1 is not true. Thus, if a point belongs to a set, that set does not automatically become its neighbourhood. For instance, take the case of $1 \in \mathbb{N}$. Then*

$$\mathbb{N} \notin \mathcal{V}(1),$$

*because there exists no ball (and therefore there exitst no radius $r > 0$) such that $B(1, r) \subset \mathbb{N}$.*

Lemma 3.2. *Let $a \in \mathbb{R}$, and consider two sets $V \subseteq \mathbb{R}$ and $W \subseteq \mathbb{R}$ such that $V \in \mathcal{V}(a)$, and $V \subseteq T$. Then $T \in \mathcal{V}(a)$ too.*

*Proof.* From the definition of a neighbourhood, for $V \in \mathcal{V}(a)$, there exists $r_V > 0$ such that

$$B(a, r_v) \subseteq V \subseteq T.$$

Hence $T \in \mathcal{V}(a)$. □

Lemma 3.3. *Let $a \in \mathbb{R}$, and consider two sets $V, T \subseteq \mathbb{R}$ such that both of them are neighbourhoods of $a$, i.e. they belong to $\mathcal{V}(a)$. Then*

$$V \cap W \in \mathcal{V}(a).$$

*Proof.* From the definition of a neighbourhood, for $V \in \mathcal{V}(a)$, there exists $r_V > 0$ such that

$$B(a, r_v) \subseteq V.$$

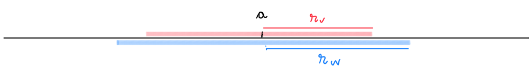
In a similar manner, $W \in \mathcal{V}(a)$, there exists $r_W > 0$ such that

$$B(a, r_w) \subseteq W.$$

Consider now $r := \min\{r_V, r_W\} > 0$. Then

$$B(a, r) \in B(a, r_V) \cap B(a, r_W) \subseteq V \cap W$$

Hence $V \cap W \in \mathcal{V}(a)$. In the following you can see an example when $r_V < r_W$.



□

The condition in Lemma 3.3 where it is required for both $V$ and $W$ to be neighbourhoods of the same point is essential, as it will be proved in the following result, where we can see, that different points can have some disjoint neighbourhoods.

LEMMA 3.4. *Let us consider two distinct real numbers $a < b \in \mathbb{R}$. Then, there exist $T \in \mathcal{V}(a)$ and $U \in \mathcal{V}(b)$ such that*

$$T \cap U = \emptyset.$$

*Proof.* Since $a < b$ we introduce the notation

$$r := b - a > 0$$

and consider the sets $T := B(a, r) = (a - r, a + r)$ and $U := B(b, r) = (b - r, b + r)$. Then $T \cap U = \emptyset$. $\square$

The previous result is a follow up of the strict separation theorem of real numbers.

REMARK 3.2. *All the above listed properties for neighbourhoods in $\mathbb{R}$ , have a similar formulation in $\overline{\mathbb{R}}$, therefore, we will omit them at this point.*

## 4. OPEN AND CLOSED SETS IN $\mathbb{R}$

All the notions presented within this section are considered to be in $\mathbb{R}$, so we omit specifying this in each statement.

DEFINITION 4.1. *Let $A \subseteq \mathbb{R}$ be a set. It is said to be* **open**, *if it is a neighbourhood for all of its points. Or, equivalently, expressed in terms of balls,*

$$\forall a \in A, \quad \exists r_a > 0 \quad s.t. \quad B(a, r_a) \subseteq A.$$

EXAMPLE 4.1. *Let $a < b \in \mathbb{R}$. Then, the following sets are open:*

- *(a,b);*
- $(-\infty, a)$*;*
- $(b, \infty)$*;*

DEFINITION 4.2. *Let $A \subseteq \mathbb{R}$ be a set. It is said to be* **closed**, *if $\mathbb{R} \backslash A$ is open.*

EXAMPLE 4.2. *Let $a < b \in \mathbb{R}$. Then, the following sets are closed:*

- $\{a\}$*;*

a,b *;*

- $(-\infty, a]$*;*
- $[b, \infty)$*;*

REMARK 4.1. a) *There is a common misunderstanding concerning the openness or the closeness o set, namely, the fact that if a certain set is not open, than it necessarily has to be closed or vice versa. This is*

**not true** *in general. There are plenty of sets, with are* **neither open, nor closed**. *For example, for* $a, b \in \mathbb{R}$, *the sets*

$$(a, b] \quad \textit{and} \quad [a, b)$$

*are neither open, nor closed.*

b) *In* $\mathbb{R}$ *there are two sets, which are* **both open and closed** *at the same time, i.e,* $\mathbb{R}$ *itself and* $\emptyset$.

In the following we present some basic properties of open sets, without proof. Those interested are encouraged to try and solve them by themselves, and in case of need, they can be encountered in [2].

Proposition 4.1. *The following statements are true:*

a) $\mathbb{R}$ *and* $\emptyset$ *are open sets.*
b) *The random reunion of open sets is open.*
c) *The random intersection of a finite number of open sets is open.*

The last statement of the previous proposition is the one that poses most problems in understanding, and it leads in general to more difficult exercises. A counterexample for this tatemes c) is the following.

Example 4.3. *For each* $n \in \mathbb{N}$, *consider the sets*

$$B_n = \left(-\frac{1}{n}, \frac{1}{n}\right).$$

*Study weather the set*

$$\bigcup_{n \in \mathbb{R}} B_n$$

*is open, closed or neither of both.*

**Solution:**

To begin with, let us start by noticing that for all $n \in \mathbb{N}$, the set $B_n$ is open, since it is of the form $a)$ in Example 5.1. The idea is that

$$B = \bigcup_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$

In order to prove this statement, we have to treat is as a set equality, and prove two inclusion. We notice that

$$\forall n \in \mathbb{N}, \quad 0 \in B_n, \quad \text{therefore} \quad 0 \in B. \tag{1}$$

In order to prove that

$$B \subseteq \{0\},$$

we proceed by contradiction, assuming that the previous statement is false. Assume thus that

$$B \not\subseteq \{0\}. \tag{2}$$

Then, according to (1) this means that there exists $b \in B$ such that $b \neq 0$. Since all the sets in the intersection are symmetric, then

$$|b| > 0 \text{ has to belong to } B \text{ as well.}$$

Hence

$$\forall n \in \mathbb{N}, \quad |b| \in B_n. \tag{3}$$

which actually means that

$$\forall n \in \mathbb{N}, \quad |b| < \frac{1}{n}, \Longleftrightarrow \forall n \in \mathbb{N}, \quad n < \frac{1}{|b|}.$$

Thus $|b| > 0$ becomes an upper bound of the set of natural numbers, which is absurd, hence, our contradiction. Thus (2) is false and the proof is complete. Thus $B = \{0\}$, which is a closed set, due to the fact that

$$\mathbb{R} \backslash \{B\} = (-\infty, 0) \cup (0, \infty)$$

is an open set, as a reunion of open sets.

The previous example provides us with a **closed set** which can be written as **an intersection of an infinite number of open sets**.

REMARK 4.2. *Random intersections of open sets are not necessarily open.*

We continue next, with the counterpart of closed sets,

PROPOSITION 4.2. *The following statements are true:*
a) $\mathbb{R}$ *and* $\emptyset$ *are closed sets.*
b) *The random intersection of closed sets is closed.*
c) *The random reunion of a finite number of closed sets is closed.*

EXAMPLE 4.4. *The set of the natural numbers is closed, i.e.*

$$\mathbb{N} \quad \textit{is closed.}$$

**Solution:** When considering the complementary set of $\mathbb{N}$, we notice that

$$\mathbb{R} \backslash \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup ... \cup (n, n+1) \cup ....$$

Thus

$$\mathbb{R} \backslash \mathbb{N} = (-\infty, 1) \cup \bigcup_{n \in \mathbb{N}} (n, n+1).$$

In light of Example 5.1, each set in the reunion is open, thus using Proposition 4.1 b), $\mathbb{R} \backslash \mathbb{N}$ is a closed set, therefore, according to the definition $N$ is a closed set.

REMARK 4.3. *Using a similar proof like the one in the previous example, one can easily prove that the set of the integer numbers is closed, i.e.*

$$\mathbb{Z} \quad \textit{is closed .}$$

Example 4.4 and Remark 4.3 provide us with examples of **random reunions of an infinite numbers of closed sets, which has the result closed**. But, they should not lead us towards the mistaken conclusion that in general, the same conclusion always holds.

EXAMPLE 4.5. *Consider the following subset of* $\mathbb{R}$*,*

$$A := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \bigcup_{n\in\mathbb{N}} \left\{ \frac{1}{n} \right\}.$$

*Study weather this set $A$ is open, close of none.*

**Solution** To begin with, it is quite clear that this set is **not open**. In order to prove this, assume by contradiction that it is open. Therefore, since $1 \in A$, we must have

$$A \in \mathcal{V}(1),$$

thus, there exists $r_1 > 0$ such that $B(1, r) \subset A$. In particular, the point $a = 1 + \frac{r}{2} \in B(1, r)$. But, $a$ cannot be written as $\frac{1}{n}$, with $n \in \mathbb{N}$, thus $a \notin A$, hence the contradiction. Thus $A \notin \mathcal{V}(1)$, so $A$ is not an open set.

We might thing that it is a closed set, since it can be seen as a random reunion of closed sets (recall that for each $n \in \mathbb{N}$, the set $\left\{\frac{1}{n}\right\}$ is closed - see Example 5.2). In order to clarify this, we need to analyse

$$R\backslash A,$$

to see weather it is open or not. Since $0 \notin A$, it is clear that

$$0 \in \mathbb{R}\backslash A.$$

Let us analyze if $\mathbb{R}\backslash A$ is a neighbourhood of 0, thus assume that

$$\mathbb{R}\backslash A \in \mathcal{V}(0). \tag{4}$$

Thus, there exists $r_0 > 0$ such that $B(0, r_a) \subset \mathbb{R}\backslash A \in \mathcal{V}(a)$, which means explicitly that

$$\forall n \in \mathbb{N}, \quad \frac{1}{n} \notin B(0, r_0). \tag{5}$$

However, since $r_0 > 0$, it holds that $\frac{1}{r_0} > 0$. According to Archimedes' theorem, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{r_0} < n_0 \Longleftrightarrow \frac{1}{n_0} < r_0 \Longleftrightarrow \frac{1}{n_0} \in B(0, r_0). \tag{6}$$

From (5) and (6) we obtain a contradiction, hence (4) is false, and thus

$$\mathbb{R}\backslash A \notin \mathcal{V}(0).$$

This is enough to prove that $\mathbb{R}\backslash A$ is not an open set, therefore $A$ is not closed. In conclusion, this set

$$\bigcup_{n\in\mathbb{N}} \left\{ \frac{1}{n} \right\} \quad \text{is neither } \textbf{open} \text{ nor } \textbf{closed},$$

regardless of the fact that it is **a reunion of closed sets.**

In light of the previous example, we may state the following remark.

REMARK 4.4. *Random reunions of closed sets are not always closed.*

We propose for the reader, the following exercise, whose solution can be deduced easily, relying the examples previously stated in this section.

EXERCIŢIUL 1. *Consider now an enhance of previous example, namely the set*

$$C := \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \{0\} \cup \bigcup_{n \in \mathbb{N}} \left\{ \frac{1}{n} \right\}.$$

*Prove that $C$ is a* **closed** *set.*

EXAMPLE 4.6. *Both the sets of rational and irrational numbers are neither open, nor closed in $\mathbb{R}$*

**Solution.**

We will only deal with $\mathbb{Q}$, since $\mathbb{R}\backslash\mathbb{Q}$ has a similar behavior.

First we prove that $\mathbb{Q}$ is not open. Assume by contradiction that it is open. Then, for $0 \in \mathbb{Q}$, this means that $\mathbb{Q} \in \mathcal{V}(0)$, thus, there exists $r_0 > 0$ such that $B(0, r_0) \in \mathbb{Q}$. But, since $0 < r_0$, according to the density property of the set of irrational numbers, there exists at least one $t \in \mathbb{R}\backslash\mathbb{Q}$ such that

$$0 < t < r_0.$$

But this would mean that $t \in \mathbb{Q} \cap \mathbb{R}\backslash\mathbb{Q} = \emptyset$. Hence, the desired contradiction. Thus, $\mathbb{Q}$ is not open.

Then, we prove that $\mathbb{Q}$ is not closed. Assume by contradiction that it is closed. Then, this means that $\mathbb{R}\backslash\mathbb{Q}$ is open. Then, for $\sqrt{2} \in \mathbb{Q}$, this means that $\mathbb{Q} \in \mathcal{V}(0)$, thus, there exists $r' > 0$ such that $B(0, r') \in \mathbb{R}\backslash\mathbb{Q}$. But, since $0 < r'$, according to the density property of the set of irrational numbers, there exists at least one $u \in \mathbb{Q}$ such that

$$\sqrt{2} < u < r'.$$

But this would mean that $u \in \mathbb{Q} \cap \mathbb{R}\backslash\mathbb{Q} = \emptyset$. Hence, the desired contradiction. Thus, $\mathbb{R}\backslash\mathbb{Q}$ is not open, therefore $\mathbb{Q}$ is not closed.

## 5. OPEN AND CLOSED SETS IN $\overline{\mathbb{R}}$

The problem of studying openness and closeness in $\overline{\mathbb{R}}$ proves to be quite delicate, since there are specific examples that do not behave as expected. Let us begin with the definitions.

DEFINITION 5.1. *Let $A \subseteq \overline{\mathbb{R}}$ be a set. It is said to be* **open***, if it is a neighbourhood for all of its points. Or, equivalently, expressed in terms of balls,*

$$\forall a \in A, \quad \exists r_a > 0 \quad s.t. \quad B(a, r_a) \subseteq A.$$

Example 5.1. *Let $a < b \in \mathbb{R}$. Then, the following sets are open:*

- *(a,b);*
- $(-\infty, a)$, $[-\infty, a)$*;*
- $(b, \infty)$, $(b, \infty]$*;*

Remark 5.1. *What is interesting in this example is that both $(-\infty, a)$ and $[-\infty, a)$ are, according to the definition,* **open sets**.

Definition 5.2. *Let $A \subseteq \overline{\mathbb{R}}$ be a set. It is said to be* **closed***, if $\overline{\mathbb{R}} \backslash A$ is open.*

Example 5.2. *Let $a < b \in \mathbb{R}$. Then, the following sets are closed:*

- $\{a\}$*;*

a,b *;*

- $[-\infty, a]$*;*
- $[b, \infty]$*;*

Remark 5.2. *In $\overline{\mathbb{R}}$, for a random $a \in \mathbb{R}$, the sets $(-\infty, a]$ and $[a, \infty)$ are neither open, nor closed.*

**Solution.**

We will address the case of $(-\infty, a]$. This set is no open, since it is clearly not a neighbourhood of $a$. Assume now that it is closed. This means that

$$\overline{\mathbb{R}} \backslash (-\infty, a] = \{-\infty\} \cup (1, \infty]$$

is an open set. However, there exists no $r > 0$ such that

$$B(-\infty, r) = [-\infty, r) \subseteq \{-\infty\} \cup (a, \infty].$$

Thus $\{-\infty\} \cup (1, \infty] \notin \mathcal{V}(-\infty)$, hence, it is not an open set.

Hence $(-\infty, a]$ is neither open, nor closed.

Example 5.3. *In $\overline{\mathbb{R}}$, the set of natural numbers is* **not closed** *(in contrast to its behaviour when just considered a subset of $\mathbb{R}$).*

**Solution.** Assume now that $\mathbb{N}$ is closed. This means that

$$\overline{\mathbb{R}} \backslash \mathbb{N}$$

is an open set, in particular, it is a neighbourhood of $\infty$. Thus there exists an $r > 0$ such that

$$B(\infty, r) = (r, \infty] \subseteq \overline{\mathbb{R}} \backslash \mathbb{N}. \tag{7}$$

Since $r > 0$, according to Archimedes Axiom, there exists an $n_r \in \mathbb{N}$ such that $r < n_r \Longrightarrow n_r \in (r, \infty]$. By using (7) this means that the natural number $n_r$ does not belong to $\mathbb{N}$, so we get to a contradiction.

Hence, $\mathbb{N}$ is not an open set (when considered in the topology on $\overline{R}$.)

Remark 5.3. *With quite a similar proof, we can state that $\mathbb{Z}$ is not a closed set in $\overline{\mathbb{R}}$.*

As easily remarkable, the change of topology induces some unnatural behaviour in certain sets. This situation seems to be extremely confusing to the first year students, who are quite unexperienced in highly theoretical proofs. This is why, when introducing these notions for the first time, the exercises are delivered just in $\mathbb{R}$, no explicit example in the topology on $\overline{\mathbb{R}}$ is considered. Should we have more hours for the seminar, such changes of topology could prove to be a useful exercise in thinking. The only thing mentioned is the form of the balls and neighbourhoods in $\overline{\mathbb{R}}$, since we further need them in characterization theorems of sequences of real numbers and functions.

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*Faculty of Mathematics and Computer Science*
*"Babeş-Bolyai" University*
*Str. Kogălniceanu, no. 1*
*400084 Cluj-Napoca, Romania*
e-mail: ancagrad@math.ubbcluj.ro