DIDACTICA MATHEMATICA, Vol. 35 (2017), Pages 45-58

PROPERTIES OF RANDOM WALKS IN DIMENSION ONE

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Abstract. In this paper we present some properties of random walks in dimension one. First, we consider properties of the random walk on the integer axis, regarding the computation of various probabilities. We present the method of counting paths, the reflection principle and some techniques to prove combinatorial formulas. Second, we consider a few properties of the random walk on the circle. Some illustrative examples are also given.

MSC 2000. 60-01, 60G17.

Key words. Random walk on the integer axis, random walk on a circle, binomial distribution, reflection principle.

1. INTRODUCTION

A random walk is a stochastic model, describing a succession of random steps on some mathematical space, such as the integer axis of numbers, equidistant points placed on the circle, the points having integer coordinates in the 2 or 3-dimensional Euclidean space, the vertices of a graph, etc. The path traced by a molecule as it travels in a liquid or a gas, the price of a fluctuating stock, the financial status of a gambler and many other similar models can be approximated by random walk models. Random walks have, for example, applications in computer science, physics, chemistry, biology, economics etc. For a detailed account of the theory and its applications we refer to the works [3, 5, 6, 9].

This paper aims to present:
• a few properties of the random walk on the integer axis and on the circle,
• the reflection principle,
• methods of computing probabilities by counting paths of the random walk represented in the Cartesian coordinate system,
• techniques to prove combinatorial formulas by using the number of paths of a random walk,
• examples such that the theory can be followed more easier.

The paper also contains an elementary proof for the result of Pólya [8], which states that a symmetric random walk along the integer axis, which starts at the origin, returns almost surely to the origin (see Theorem 1). The proof avoids the classic method of using power series expansions of certain functions, but uses the method of equating the coefficients of polynomial functions and convergence properties of sequences.

The results of this paper can be understood by scholars or students, teachers or researchers, who have basic notions of probability theory. The probabilistic notions and notations used within this paper are briefly presented in the Appendix section.

The following notation is very often used throughout this paper:

$$C_n^k = \frac{n!}{k!(n-k)!}, \text{ for each } n \in \mathbb{N} \text{ and } k \in \{0, 1, \dots, n\}, \text{ where } 0! = 1.$$

2. RANDOM WALK ON A LINE

Let $p \in (0, 1)$. A random walk, which starts at some $k \in \mathbb{Z}$, takes place along the **axis** of numbers, from one integer to another one, in the following way: For each step, we go with probability p to the first larger integer and with probability $1 - p$ to the first smaller integer (see Figure 1). Each step is independent of the previous one. If $p = \frac{1}{2}$, then the random walk is called symmetric.

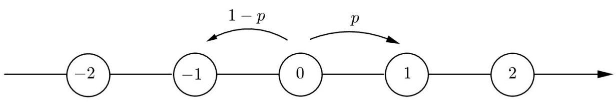

Fig. 1 – Random walk on the integer axis starting at 0.

Denote by S_n the position of the random walk on the integer line after $n \in \mathbb{N}$ steps, which starts at some $k \in \mathbb{Z}$. Note, that S_n is a discrete random variable taking values in the set $\{k-n, \dots, k-1, k, k+1, \dots, k+n\}$. Obviously, $S_0 = k$, $P(S_{n+1} = j+1 | S_n = j) = p$ and $P(S_{n+1} = j-1 | S_n = j) = 1 - p$, for $j \in \{k-n, \dots, k-1, k, k+1, \dots, k+n\}$.

Let $(X_n)_{n \geq 1}$ be a sequence of independent discrete random variables having the distribution

$$P(X_n = 1) = p \text{ and } P(X_n = -1) = 1 - p, \qquad \text{for each } n \geq 1.$$

We have

$$S_n - S_0 = X_1 + X_2 + \dots + X_n, \qquad \text{for each } n \geq 1.$$

Proposition 1. For $p \in (0, 1)$, $n \in \mathbb{N}$ and $j \in \mathbb{Z}$ with $|j| \leq n$ given, the probability to end up in the integer number j after a random walk of n steps, which starts at 0, is

$$P(S_n = j) = C_n^{\frac{n+j}{2}} p^{\frac{n+j}{2}} (1-p)^{\frac{n-j}{2}},$$

if j and n have the same parity, and the probability to end up in the integer number j is $P(S_n = j) = 0$, if j and n have opposite parity.

j	-4	-3	-2	-1	0	1	2	3	4
$P(S_0 = j)$					1				
$P(S_1 = j)$				$\frac{1}{2}$		$\frac{1}{2}$			
$P(S_2 = j)$			$\frac{1}{2^2}$		$\frac{2}{2^2}$		$\frac{1}{2^2}$		
$P(S_3 = j)$		$\frac{1}{2^3}$		$\frac{3}{2^3}$		$\frac{3}{2^3}$		$\frac{1}{2^3}$	
$P(S_4 = j)$	$\frac{1}{2^4}$		$\frac{4}{2^4}$		$\frac{6}{2^4}$		$\frac{4}{2^4}$		$\frac{1}{2^4}$

Table 1 – Probabilities for a symmetric random walk on the integer line.

Proof. Denote by n_R the number of steps taken by the random walk rightward and by n_L the number of steps taken leftward. Obviously, $n_R, n_L \in \{0, 1, \ldots, n\}$ and we have

$$n_R + n_L = n \text{ and } n_R - n_L = j. \tag{1}$$

If j and n have the same parity, we have

$$n_R = \frac{n+j}{2} \text{ and } n_L = \frac{n-j}{2}.$$

If j and n do not have the same parity, there are no integer numbers n_R and n_L satisfying (1), in fact there exists no random walk of n steps that ends up in j.

Every random walk of n steps can be identified with n Bernoulli trials (see Proposition 6). Hence, the probability to make n_R steps rightward and $n_L = n - n_R$ steps leftward is $C_n^{n_R} p^{n_R} (1-p)^{n_L}$.

Therefore, if j and n have the same parity, then the probability to end up in j is $C_n^{\frac{n+j}{2}} p^{\frac{n+j}{2}} (1-p)^{\frac{n-j}{2}}$, and, if j and n have opposite parity, then the probability to end up in j is 0. □

Example 1. For a random walk on the integer axis, which starts at 0, we compute:

(1) $P(S_6 = S_{10}) = P(X_7 + X_8 + X_9 + X_{10} = 0) = 6p^2(1-p)^2$;

(2) $P(S_5 = -1, S_{11} = 3) = P(S_5 = -1, S_{11} - S_5 = 4)$

$= P(X_1 + X_2 + X_3 + X_4 + X_5 = -1, X_6 + X_7 + X_8 + X_9 + X_{10} + X_{11} = 4)$

$= C_5^2 p^2 (1-p)^3 \cdot C_6^5 p^5 (1-p) = 60p^7(1-p)^4$;

(3) $P(S_6 = S_{11}) = P(X_7 + X_8 + X_9 + X_{10} + X_{11} = 0) = 0.$

Representation of a random walk on the integer line: We represent a random walk as a continuous path, or polygonal line, in the Cartesian coordinate system, where the horizontal axis represents the discrete (time) steps and the vertical axis represents the position of the random walk, i.e., the point (j, S_j) indicates that after j steps the random walk is at the position S_j on the integer line. Let $n \in \mathbb{N}^*$. We plot the points $(0, S_0), (1, S_1), \dots, (n, S_n)$, and then for each $j \in \{0, \dots, n-1\}$, we connect (j, S_j) and $(j+1, S_{j+1})$ with a straight line segment (see [3], [6]). We shall identify each random walk of n steps along the integer axis, which starts at $S_0 = k$, with the *path* that starts at $(0, k)$ and in each step connects the points (j, S_j) and $(j+1, S_{j+1})$, $j \in \{0, \dots, n-1\}$ (see Figure 2 for $k = 2$ and $n = 7$). For $n = 0$ the *path* is just the point $(0, k)$.

We observe that $S_n - S_0$ denotes the difference between the number of rightward steps and the number of leftward steps after n steps.

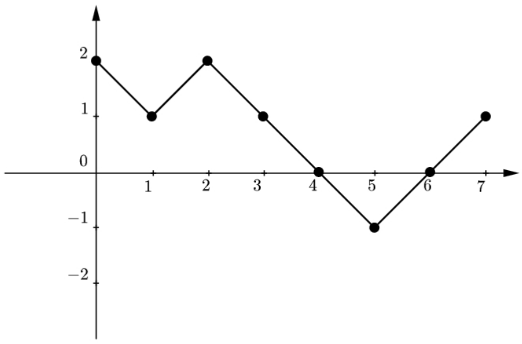

Fig. 2 – A path of a random walk.

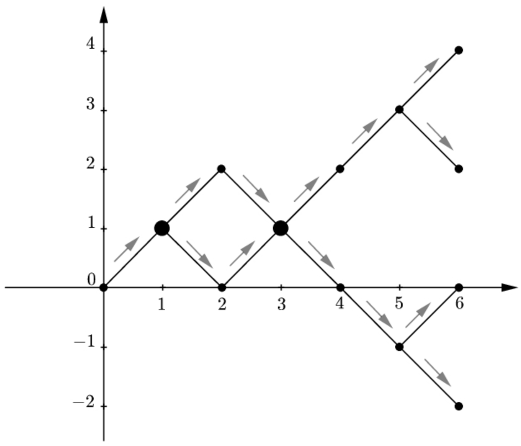

Fig. 3 – For a 6 steps symmetric random walk on the integer line, starting at 0: 8 paths connecting the points with coordinates $(1, 1)$ and $(3, 1)$.

Example 2. The probability that after 6 steps a symmetric random walk on the integer line, which starts at 0, passed the number 1 exactly 2 times is $\dfrac{16}{2^6}$. To compute this we use the representation with paths and count the number of all possible paths: there are 8 paths connecting the points with coordinates $(1,1)$ and $(3,1)$ (see Figure 3), 4 paths connecting the points with coordinates $(1,1)$ and $(5,1)$ and 4 paths connecting the points with coordinates $(3,1)$ and $(5,1)$.

2.1. **The reflection principle.** In the following, we consider paths that represent random walks on the integer line.

Proposition 2. Let $k,m\in\mathbb{Z}$ and $l\in\mathbb{N}$.
(i) A necessary and sufficient condition for the existence of a path that starts at $(0,k)$ and ends at (l,m) is: $|m-k|\le l$ and $l+m-k$ is an even number.
(ii) Suppose that $|m-k|\le l$ and $l+m-k$ is an even number. Then, the number of paths that start at $(0,k)$ and end up at (l,m) is $C_l^{\frac{l+m-k}{2}}$.

Proof. (i) Assume that there exists a path that starts at $(0,k)$ and ends at (l,m), i.e. the random walk starts at position k, makes l steps and ends up at the position m: Denote by n_R the number of steps taken by the random walk rightward and by n_L the number of steps taken leftward. Obviously, $n_R,n_L\in\{0,1,\dots,l\}$ and we have

$$n_R+n_L=l \text{ and } n_R-n_L=m-k. \tag{2}$$

Hence,

$$2n_R=l+m-k \text{ and } 2n_L=l-m+k,$$

which implies that $|m-k|\le l$ and $l+m-k$ is an even number.

Assume now that $|m-k|\le l$ and $l+m-k$ is an even number. Consider the random walk that starts at position k and makes $n_R=\frac{l+m-k}{2}$ steps to the right and $n_L=\frac{l-m+k}{2}$ to the left. Obviously the corresponding path of this random walk will start at $(0,k)$ and end up at (l,m).
(ii) Using the results of (i) we have that the number of paths of length l with n_R steps rightward and $n_L=l-n_R$ steps leftward is $C_l^{n_R}=C_l^{\frac{l+m-k}{2}}$ (see Proposition 6). $\square$

The following result, called the *reflection principle*, is a useful tool in the study of random walks along the axis (see [1], [6, Section 10.3], [3, Section III.1]).

Proposition 3. Let $k,l,m\in\mathbb{N}$. There is a one to one correspondence between:
- the paths that start at $(0,k)$, end at (l,m), and cross or touch the x-axis;
- the paths that start at $(0,-k)$ and end at (l,m).

Proof. The case $k=0$ is trivial. Suppose that $k\ge 1$. Let Γ be a path that starts at $(0,k)$, ends at (l,m), and crosses or touches the x-axis. Let $n\in\{1,\dots,l\}$ be the smallest integer such that $(n,0)\in\Gamma$. We observe that Γ is the union of a path Γ' that starts at $(0,k)$ and ends at $(n,0)$ and a path Γ'' that starts at $(n,0)$ and ends at (l,m). Let $\widetilde{\Gamma}'$ be the reflection of Γ' across the x-axis (see Figure 4). We associate to Γ the path $\widetilde{\Gamma}$ that is the union of $\widetilde{\Gamma}'$ and Γ''. It is easy to verify that this correspondence is one to one.

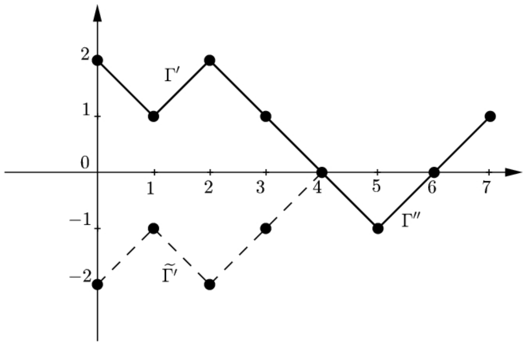

Fig. 4 – The reflection principle applied for a path.

$\square$

In the following we present some applications of the *reflection principle* (see [3], [6]).

Remark 1. Let $k, l, m \in \mathbb{N}$. In view of the reflection principle, we deduce that for a symmetric random walk

$$P(S_l = m \text{ and } \exists\ n \in \{0, 1, \dots, l\} : \ S_n = 0 | S_0 = k) = P(S_l = m | S_0 = -k).$$

Corollary 1. Let $n \in \mathbb{N}^*$. The number of paths that start at $(0,0)$, end at $(2n, 0)$ and don't cross or touch the x-axis, apart from the starting point and the ending point, is $\frac{2}{n} C_{2(n-1)}^{n-1}$.

Proof. We note that the number of paths that start at $(0,0)$, end at $(2n, 0)$ and don't cross or touch the x-axis, apart from the starting point and the ending point, is twice the number of paths that start at $(1,1)$, end at $(2n-1, 1)$ and don't cross or touch the x-axis, which is the same as the number of paths that start at $(0,1)$, end at $(2n-2, 1)$ and don't cross or touch the x-axis. By the *reflection principle*, we deduce that the number of paths that start at $(0,1)$, end at $(2n-2, 1)$ and cross or touch the x-axis is equal to the number of paths that start at $(0,-1)$ and end at $(2n-2, 1)$, which is C_{2n-2}^{n} (by Proposition 2). Since the number of paths that start at $(0,1)$ and end at $(2n-2, 1)$ is C_{2n-2}^{n-1} (by Proposition 2), we have that the number of paths that start at $(1,1)$, end at $(2n-1, 1)$ and don't cross or touch the x-axis is $C_{2n-2}^{n-1} - C_{2n-2}^{n} = \frac{1}{n} C_{2(n-1)}^{n-1}$. $\square$

Corollary 2. Let $n \in \mathbb{N}^*$. Then

$$\sum_{k=1}^{n} \frac{2}{k} C_{2(k-1)}^{k-1} C_{2(n-k)}^{n-k} = C_{2n}^{n}.$$

Proof. The number of paths that start at $(0,0)$ and end at $(2n,0)$ is C_{2n}^{n} (by Proposition 2). In the following, we shall count the number of these paths in a different way. Let $k \in \{1,\ldots,n\}$. By Corollary 1, we have that the number of paths that start at $(0,0)$, end at $(2k,0)$ and don't cross or touch the x-axis, apart from the starting point and the ending point, is $\frac{2}{k}C_{2(k-1)}^{k-1}$. The number of paths that start at $(2k,0)$ and end at $(2n,0)$ is the same as the number of paths that start at (0,0) and end at $(2n-2k,0)$. This number is $C_{2(n-k)}^{n-k}$ (by Proposition 2). We deduce that the number of paths that start at $(0,0)$, end at $(2n,0)$ and cross or touch the x-axis for the first time at $(2k,0)$ is $\frac{2}{k}C_{2(k-1)}^{k-1}C_{2(n-k)}^{n-k}$. Hence the number of paths that start at $(0,0)$ and end at $(2n,0)$ is $\sum_{k=1}^{n}\frac{2}{k}C_{2(k-1)}^{k-1}C_{2(n-k)}^{n-k}$. □

2.2. Return to the origin. Let $n \in \mathbb{N}^*$. The number of all possible symmetric random walks of $2n$ steps that start at 0 is 2^{2n}. Taking into account the identification between the symmetric random walks and the paths, we have, by Corollary 1, that the probability that a symmetric random walk of $2n$ steps, which starts at 0, visits 0 again only at the last step is $\frac{1}{n2^{2n-1}}C_{2(n-1)}^{n-1}$.

Let $N \in \mathbb{N}^*$. We denote

$$p_N = \sum_{n=1}^{N}\frac{1}{n2^{2n-1}}C_{2(n-1)}^{n-1}.$$

For $n \in \{0,\ldots,N\}$ consider A_{2n} to be the event that a symmetric random walk starting at the origin returns for the first time to the origin at the step $2n$. Obviously, these events are pairwise disjoint and

$$P(A_{2n}) = \frac{1}{n2^{2n-1}}C_{2(n-1)}^{n-1}.$$

By the properties of the probability function (see Definition 2)

$$\begin{aligned} p_N &= \sum_{n=1}^{2N}\frac{1}{n2^{2n-1}}C_{2(n-1)}^{n-1} = \sum_{n=1}^{2N}P(A_{2n}) = P(\bigcup_{n=1}^{2N}A_{2n}) \\ &= P(\text{"a symmetric random walk of } 2N \text{ steps, starting at 0, visits 0 again"}). \end{aligned}$$

Hence, $(p_N)_{N\geq 1}$ is an increasing sequence of numbers in the interval $(0,1)$. Moreover, by the convergence properties of the probability function (see Proposition 5) we have

$$\begin{aligned} \lim_{N\to\infty} p_N &= \lim_{N\to\infty} P(\bigcup_{n=1}^{2N}A_{2n}) = P(\bigcup_{n=1}^{\infty}A_{2n}) \\ &= P(\text{"a symmetric random walk, starting at 0, visits 0 again"}). \end{aligned}$$

We denote

$$\lim_{N\to\infty} p_N = \hat{p}.$$

In the following, we present an elementary proof for the result of Pólya [8], which states that a symmetric random walk along the integer axis, which starts at the origin, returns almost surely to the origin (cf. [2], [4]). Instead of the classic method of power series expansions of certain functions, we use the method of equating the coefficients of polynomial functions and convergence properties of sequences.

Theorem 1. The probability that a symmetric random walk along the integer axis, which starts at the origin, returns to the origin is 1, i.e.

$$\lim_{N\to\infty} p_N = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n2^{2n-1}} C_{2(n-1)}^{n-1} = 1.$$

Proof. For every $N \in \mathbb{N}^*$, we define

$$F_N(x) = \sum_{k=1}^{N} C_{2(k-1)}^{k-1} \frac{x^k}{k}, \quad x \in \mathbb{R}. \tag{3}$$

Let $n, N \in \mathbb{N}^*$ be such that $N \geq n$. We note that the coefficient of x^n in $2F_N(x)F_N'(x)$ is $2\sum_{k=1}^{n} C_{2(k-1)}^{k-1} \frac{1}{k} C_{2(n-k)}^{n-k}$, which is equal to C_{2n}^{n}, by Corollary 2. By integrating, we deduce that the coefficient of x^{n+1} in $F_N^2(x)$ is $C_{2n}^{n} \frac{1}{n+1}$. On the other hand, we have that the coefficient of x^{n+1} in $F_N^2(x)$ is $\sum_{k=1}^{n} C_{2(k-1)}^{k-1} \frac{1}{k} C_{2(n-k)}^{n-k} \frac{1}{n-k+1}$.

In view of the above, we deduce that

$$\sum_{k=1}^{n} C_{2(k-1)}^{k-1} \frac{1}{k} C_{2(n-k)}^{n-k} \frac{1}{n-k+1} = C_{2n}^{n} \frac{1}{n+1}, \text{ for all } n \in \mathbb{N}^*. \tag{4}$$

Let $N \in \mathbb{N}$ be such that $N \geq 2$. Using again the above discussion, we have that

$$F_N^2(x) = F_{N+1}(x) - x + x^{N+2} R_N(x), \quad x \in \mathbb{R},$$

where R_N is a polynomial function. In the following, we shall take a closer look at the coefficients in $x^{N+2}R_N(x)$. For every $m \in \mathbb{N}$ with $2 \leq m \leq N$, the coefficient of x^{N+m} in $F_N^2(x)$ is

$$\sum_{k=m}^{N} C_{2(k-1)}^{k-1} \frac{1}{k} C_{2(N+m-k-1)}^{N+m-k-1} \frac{1}{N+m-k}$$

which is positive and bounded above by

$$\sum_{k=1}^{N+m-1} C_{2(k-1)}^{k-1} \frac{1}{k} C_{2(N+m-k-1)}^{N+m-k-1} \frac{1}{N+m-k} = C_{2(N+m-1)}^{N+m-1} \frac{1}{N+m},$$

where the above equality follows from (4). Taking into account (3), we deduce that

$$0 \leq F_N^2(x) - F_{N+1}(x) + x \leq F_{2N}(x) - F_{N+1}(x), \text{ for all } x \geq 0. \tag{5}$$

Recall that

$$p_N = \sum_{n=1}^{N} \frac{1}{n2^{2n-1}} C_{2(n-1)}^{n-1},$$

and thus $p_N = 2F_N(\frac{1}{4})$. Since $(p_N)_{N\geq 1}$ converges to $\hat{p}$, we deduce, by taking $x = \frac{1}{4}$ and $N \to \infty$ in (5), that $\left(\frac{\hat{p}}{2}\right)^2 - \frac{\hat{p}}{2} + \frac{1}{4} = 0$. Thus $\hat{p} = 1$. □

Remark 2. For every $N \in \mathbb{N}^*$, let F_N be given by (3). Let $x \geq 0$. We observe that $(F_N(x))_{N\geq 1}$ is an increasing sequence of non-negative numbers. Let $F(x) = \lim_{N\to\infty} F_N(x) \in [0, \infty]$. If $F(x) < \infty$, then, by taking $N \to \infty$ in (5), we deduce that $F(x)$ is a solution of the following equation

$$t^2 - t + x = 0.$$

Since $\Delta = 1-4x$, this equation has a solution only if $x \leq \frac{1}{4}$. Hence, for $x > \frac{1}{4}$, we have $F(x) = \infty$. For $x \in [0, \frac{1}{4}]$, we note that $F(x) \leq F(\frac{1}{4}) = \frac{1}{2}$ (see the proof of Theorem 1), hence $F(x) = \frac{1-\sqrt{1-4x}}{2}$. Thus, we have

$$\lim_{N\to\infty} \sum_{k=1}^{N} C_{2(k-1)}^{k-1} \frac{x^k}{k} = \begin{cases} \frac{1-\sqrt{1-4x}}{2}, & \text{for } x \in [0, \frac{1}{4}], \\ \infty, & \text{for } x \in (\frac{1}{4}, \infty). \end{cases}$$

Remark 3. Let $p \in (0, 1)$. In the following, we consider the random walks that start at 0 and, for each step, move rightward with probability p and leftward with probability $1-p$. Let $n \in \mathbb{N}^*$. By Corollary 1, there are $\frac{2}{n} C_{2(n-1)}^{n-1}$ possible random walks of $2n$ steps that end at 0 and don't visit 0 except the starting point and the ending point. Each one of these random walks takes place with probability $p^n(1-p)^n$, because n rightward steps and n leftward steps are needed in order to end the walk of $2n$ steps at 0. Therefore, the probability that a random walk of $2n$ steps visits 0 only at the last step is $\frac{2}{n} C_{2(n-1)}^{n-1} p^n (1-p)^n$.

For every $N \in \mathbb{N}^*$, let

$$q_N = 2F_N(p(1-p)) = \sum_{k=1}^{N} \frac{2}{k} C_{2(k-1)}^{k-1} p^k (1-p)^k,$$

where F_N is given by (3). Taking into account the above and adapting the discussion at the beginning of Section 2.2, we deduce that, for every $N \in \mathbb{N}^*$, the probability that a random walk of $2N$ steps, which starts at 0, visits 0 again is q_N. Since $p(1-p) \in (0, \frac{1}{4}]$, it follows (by similar arguments as in Remark 2) that

$$\lim_{N\to\infty} q_N = 2F(p(1-p)) = 1 - \sqrt{1 - 4p(1-p)}.$$

In view of the above, we have

$$\lim_{N\to\infty} q_N = 1 - |2p-1| = \begin{cases} 2p, & \text{for } p \in (0, \frac{1}{2}] \\ 2(1-p), & \text{for } p \in (\frac{1}{2}, 1). \end{cases}$$

The same result can be obtained by using power series expansions (see e.g. [2], [4]).

In particular, a random walk along the integer axis, which starts at the origin, returns to the origin with probability 1 if and only if $p = \frac{1}{2}$ (i.e. the random walk is symmetric).

3. RANDOM WALK ON A CIRCLE

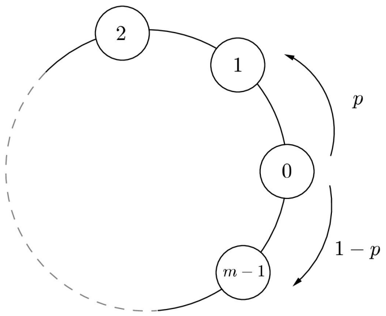

Fig. 5 – Random walk on the circle.

In the following we study a random walk that takes place along a **circle**: Let $p \in (0,1)$, $m \in \mathbb{N}^*$ be given and let the integers $0, 1, \ldots, m-1$ be placed equidistantly on a circle (anticlockwise). A random walk takes place along this circle, from one integer to another one, in the following way: The walk starts at 0. For each step, the random walk moves anticlockwise with probability p to the closest integer and clockwise with probability $1-p$ to the closest integer, see Figure 5.

Denote by Z_n the position of the random walk on the circle after $n \in \mathbb{N}$ steps. Note, that Z_n is a discrete random variable taking values in the set $\{0, 1, \ldots, m-1\}$ and obviously, $Z_0 = 0$.

Proposition 4. For $p \in (0,1)$, $n \in \mathbb{N}$, $m \in \mathbb{N}^*$ and $j \in \{0, 1, \ldots, m-1\}$ given, the probability to end up in the position j after a random walk of n steps is given by

$$P(Z_n = j) = \sum_{l \in I_j} C_n^l p^l (1-p)^{n-l},$$

where

$$I_j = \{l \in \{0, 1, \ldots, n\} : 2l - n \ (\mathrm{mod}\ m) = j\}$$

and $P(Z_n = j) = 0$, if $I_j = \emptyset$.

j	0	1	2	3
$P(S_0 = j)$	1			
$P(S_1 = j)$		$\frac{1}{2}$		$\frac{1}{2}$
$P(S_2 = j)$	$\frac{2}{2^2}$		$\frac{2}{2^2}$	
$P(S_3 = j)$		$\frac{4}{2^3}$		$\frac{4}{2^3}$
$P(S_4 = j)$	$\frac{8}{2^4}$		$\frac{8}{2^4}$	

Table 2 – Probabilities for a symmetric random walk on the circle with $m = 4$.

Proof. The random walk along the circle can be identified with the the random walk along the axis, described in Proposition 1, by replacing each integer of the axis with the corresponding remainder of the division by m. Hence, the random walk can end up only at one of the following integers: $\{2l - n \pmod m) : l \in \{0, 1, \ldots, n\}\}$. Therefore, the probability to end up in $j \in \{0, 1, \ldots, m - 1\}$ is $\sum_{l \in I_j} C_n^l p^l (1-p)^{n-l}$, where

$$I_j = \{l \in \{0, 1, \ldots, n\} : 2l - n \pmod m) = j\}$$

and the above sum is zero, if $I_j = \emptyset$. $\square$

Example 3. The probability that after 6 steps a symmetric random walk on the circle with $m = 4$, passed the number 1 exactly 2 times is $\frac{24}{2^6}$. The picture of all such paths starting from 0 and moving in the first step anticlockwise is given in Figure 6. There are 16 such paths. For example, such a random walk passes successively the points: 0, 1, 2, 3, 2, 1, 2. Similarly, one can count that the number of paths starting from 0, moving in the first step clockwise and passing the number 1 exactly 2 times is 8.

4. APPENDIX

In this section we briefly present the probabilistic notions used within this paper (for more details see, e.g. [3, 7]):

Definition 1. A collection $\mathcal{K}$ of events from the sample space Ω (i.e., $\mathcal{K} \subset \mathcal{P}(\Omega)$) is said to be a **$\sigma$-field**, if it satisfies the following conditions:
(1) $\mathcal{K} \neq \emptyset$;
(2) if $A \in \mathcal{K}$, then for its complement $\bar{A} = \Omega \setminus A$ we have $\bar{A} \in \mathcal{K}$;
(3) if $A_n \in \mathcal{K}$, for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{K}$.

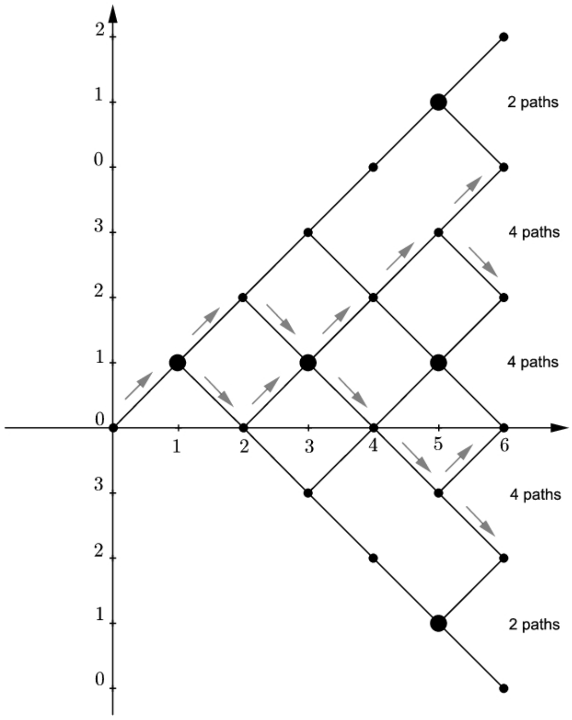

Fig. 6 – For a symmetric random walk on the circle ($m = 4$): 16 paths starting at 0 and moving anticlockwise for the first step, such that the number 1 is passed exactly 2 times.

We consider an experiment whose outcomes are finite and equally likely. Then, the **probability** that an event $A \in \mathcal{K}$ will occur is the number

$$P(A) = \frac{\text{number of outcomes favorable for the occurrence of A}}{\text{number of all possible outcomes during the experiment}}.$$

This is the classic interpretation of the probability of an event. The axiomatic definition of the probability is the following:

Definition 2. Let $\mathcal{K}$ be a σ-field in Ω. A mapping $P : \mathcal{K} \to \mathbb{R}$ is called **probability** if it satisfies the following axioms:
(1) $P(\Omega) = 1$;
(2) $P(A) \geq 0$ for every $A \in \mathcal{K}$;
(3) for any sequence $(A_n)_{n\geq 1}$ of pairwise disjoint events from $\mathcal{K}$ (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$) it holds

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} P(A_n).$$

The triplet $(\Omega, \mathcal{K}, P)$ consisting of a sample space Ω, a σ-field $\mathcal{K}$ and a probability P is called **probability space**.

Let $(\Omega, \mathcal{K}, P)$ be a probability space.

PROPOSITION 5. If $(A_n)_{n\geq 1}$ is an increasing sequence of events from $\mathcal{K}$, i.e., $A_n \subseteq A_{n+1}$ for each $n \geq 1$, then

$$\lim_{n\to\infty} P(A_n) = P(\bigcup_{n=1}^{\infty} A_n).$$

DEFINITION 3. Let $A, B \in \mathcal{K}$. The **conditional probability** of A **given** B is $P(\cdot|B) : \mathcal{K} \to \mathbb{R}$, defined by

$$P(A|B) = \frac{P(A\cap B)}{P(B)},$$

provided $P(B) > 0$.

DEFINITION 4. The events $A_1, \ldots, A_n \in \mathcal{K}$ are said to be **independent events** if

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdot \cdots \cdot P(A_n).$$

A sequence $(A_n)_{n\geq 1}$ of events from $\mathcal{K}$ is called **sequence of independent events** if

$$P(A_{i_1} \cap \cdots \cap A_{i_m}) = P(A_{i_1}) \ldots P(A_{i_m})$$

for each finite subset $\{i_1, \ldots, i_m\} \subset \mathbb{N}^*$.

Discrete random variables $X : \Omega \to \{x_i : i \in I\}$ are described by their values and probabilities

$$X \sim \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i\in I}, \text{ where } p_i = P(X = x_i),\ p_i \in (0,1] \text{ for each } i \in I \text{ and } \sum_{i\in I} p_i = 1.$$

DEFINITION 5. Let $X_1, \ldots, X_n$ be discrete random variables. They are **independent** if and only if for all $x_1, \ldots, x_n \in \mathbb{R}$ it holds

$$P(X = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdot \ldots \cdot P(X_n = x_n).$$

$(X_n)_{n\geq 1}$ is a **sequence of independent random variables** if for each finite subset $\{i_1, \ldots, i_m\} \subset \mathbb{N}^*$ the random variables $X_{i_1}, \ldots, X_{i_m}$ are independent.

Repeated independent trials of an experiment, such that there are only two possible outcomes for each trial and their probabilities remain the same throughout the trials are called **Bernoulli trials**. We denote the two possible outcomes by A,"success", having the probability $p \in [0, 1]$, and respective its complement $\bar{A}$, "failure", with probability $1 - p$.

We are interested in the total number of successes produced in n Bernoulli trials, but not in their order.

PROPOSITION 6. Given n Bernoulli trials with the probability p of success and probability $1-p$ of failure, then the probability to occur exactly k successes is $C_n^k p^k (1-p)^{n-k}$.

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