DIDACTICA MATHEMATICA, Vol. 34(2016), pp. 1–7

JULIA ROBINSON AND HILBERT'S TENTH PROBLEM

Mira-Cristiana Anisiu

Abstract. One of the solved Hilbert's problems stated in 1900 at the International Congress of Mathematicians in Paris is: *Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.* Julia Robinson (1919-1985) had a basic contribution to its negative solution, completed by Yuri Matijasevich. Her passion for Mathematics allowed her to become a professor at UC Berkeley and the first woman president of the American Mathematical Society. She firmly encouraged all the women who have the ability and the desire to do mathematical research to fight and support each other in order to succeed.

Dedicated to Anca Căpăţînă, forced to retire in February 2016 (five years earlier than male researchers) and rewarded with the Spiru Haret Prize of the Romanian Academy in December of the same year.

MSC 2000. 01-01, 01A60, 11U05
Key words. Diophantine equations, Hilbert tenth problem

1. DIOPHANTINE EQUATIONS

Diophantine equations are named after the Greek mathematician Diophantus (200 AD - 284 AD), born in Alexandria, Egypt. In his *Arithmetica*, a treatise of several books, he studied about 200 equations in two or more variables with the restriction that the solutions be rational numbers.

The simplest Diophantine equations are the two-variable linear ones, which are of the form

$$ax + by = c, \tag{1}$$

with a, b and c integers, and for which the variables x and y can only have integer values. The problem is to determine when the equation (1) has a solution, and to solve it when possible.

It is obvious that there are values of the parameters a, b and c for which the equation (1) has no solution. For example, if $a = 2$, $b = 4$ and $c = 3$ the left hand side is even and the right hand side is odd. In fact, it can be shown that (1) has no solution if c is not divisible by the greatest common divisor of a, b, denoted by (a, b); when c is divisible by (a, b), the equation has always a solution x_0, y_0, and the totality of solutions is given by

$$x = x_0 + kb/\left(a, b\right), \ y = y_0 + ka/\left(a, b\right),$$

where k is an integer.

Another well known Diophantine equation, quadratic of three variables, is

$$x^2 + y^2 = z^2. \tag{2}$$

Again one looks for integer solutions. These solutions are often known as Pythagorean triples, since a geometric interpretation is that of the lengths of the sides of a right triangle, and the expression in (2) is the Pythagorean Theorem. It can be shown that all Pythagorean triples are of the form

$$x = k\left(r^2 - s^2\right),\ y = 2krs,\ z = k\left(r^2 + s^2\right),$$

where $r > s > 0$ and r, s, and k are integers.

Actually these two problems were already known by the Babylonians. Some of the most famous problems in number theory are Diophantine equations posed by mathematicians living much later. One of them is Fermat's Last Theorem stating that the equation

$$x^n + y^n = z^n \tag{3}$$

has no solution for $n > 2$. The history of this problem from its beginnings to the solution given by Andrew Wiles is the subject of the book [15].

Diophantine equations are equations of polynomial expressions for which rational or integer solutions are sought.

2. HILBERT'S TENTH PROBLEM

Hilbert's tenth problem was stated in his 1900 lecture to the International Congress of Mathematicians in Paris, in his famous programme containing 23 problems. A book that addresses to both mathematical and nonmathematical readers, and presents the story of all of Hilbert problems is [16].

The statement of the tenth problem is as follows ([6]):

Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.

Rational integers are simply *integers*, namely $0, \pm1, \pm2, \ldots$. In fact, Hilbert was asking for an algorithm which decides whether a given Diophantine equation with integer coefficients has a solution in integers.

The problem was solved in the negative, on the base of the results of Julia Robinson, by the young Russian mathematician Yuri Matijasevich. Later on, they began a fruitful cooperation.

In 1971, at a conference in Bucharest, Robinson gave the lecture *Solving diophantine equations* ([14]) in which she indicated directions for continuing to study Diophantine equations following the negative solution to Hilbert's tenth problem. At that conference she met for the first time Yuri Matijasevich, who also gave a lecture ([9]).

Julia Robinson gave in 1975 a short description of the solution of Hilbert's tenth problem ([11]):

"The answer lies in a branch of mathematics called *recursion theory* which was developed during the 1930s by several mathematicians: Church, Godel, Kleene, Post in the United States, Herbrand in France, Turing in England, Markov in the USSR, etc. The method of proof is based on the fact that there is a Diophantine equation say $P(x, y, z, \ldots, w) = 0$ such that the set S of all values of x in all the solutions of $P = 0$ is too complicated a set to be calculated by any method whatever. If we had a method which would tell us whether $P(a, y, z, \ldots, w) = 0$ has a solution for a given value of a, then we would have a method of calculating whether a belongs to the set S, and this is impossible."

One of the corollaries of the negative solution of Hilbert's tenth problem is that there is a constant N such that, given a Diophantine equation with any number of parameters and in any number of unknowns, one can effectively transform this equation into another with the same parameters but in only N unknowns such that both equations are solvable or unsolvable for the same values of the parameters. In 1970, Matijasevich reported that N could be taken equal to 200. This estimate was very rough. By corresponding with Julia Robinson, he succeeded to reduce N to 33, then to 26 and to 14. They wrote a joint paper in Russian, in 1974.

3. JULIA ROBINSON

Fortunately, facts from the life and scientific activity of Julia Robinson are recorded in complete biographies written by her sister ([11]) or by Feferman ([4]), who edited her collected works ([5]). Julia Robinson was born in St. Louis, Missouri on December 8, 1919. Her parents were Helen Hall and Ralph Bowers Bowman, and she had an elder sister, Constance. Her father owned a small company and Helen had been a primary school teacher before her marriage. Helen died when Julia was two years old. Ralph remarried a year later, retired from his business and moved to Arizona and then to San Diego. Another daughter, Billie, was born a few years later.

Julia was very ill during her childhood, having scarlet fever at nine and then rheumatic fever, with lifelong consequences for her health. She left school for two years and, after a year of tutoring, she resumed classes in San Diego. She was the only girl in her mathematics class and in her physics class, and graduated with awards in all the sciences (except chemistry, which she had not taken). She received the Bausch-Lomb medal for all-round excellence in mathematics and science. She entered San Diego State College to study mathematics. One year later, in September 1937 Ralph Bowman committed suicide, after his retirement savings had been wiped out by the 1929 crash and the subsequent depression. The family moved to a small apartment and

an aunt provided the funds which allowed Julia and Constance to finish the college.

In 1939, Julia transferred to the University of California at Berkeley (UC Berkeley) as her interest shifted to research mathematics. She took five mathematics courses, including one in number theory taught by Raphael M. Robinson. Raphael and Julia began going for walks together, often discussing mathematics. She received her Bachelor of Arts (BA) degree in 1940 and began graduate studies. She also obtained a part-time position as an assistant of professor Jerzy Neyman in statistics. In 1941 she was awarded her Master of Arts (MA).

She married Raphael Robinson (November 2, 1911 – January 27, 1995) on December 22, 1941, and they wanted to have a family. Unfortunately, after a miscarriage, the doctors strongly advised Julia against becoming pregnant again, due to her serious heart problems. After their marriage nepotism rules prohibited her from teaching as a graduate assistant in Berkeley's mathematics department, Raphael being on the mathematics staff. During World War II Julia worked in Neyman's UC Berkeley Stat Lab.

In 1947, she began to work for her doctorate with the famous logician Alfred Tarski, who came to Berkeley in 1942, and had a major role in Julia's scientific development. She received her Ph.D. in 1948, and the results of the thesis were published in the paper [13]. Her thesis *Definability and decision problems in arithmetic* showed that the notion of an integer can be defined arithmetically in terms of the notion of a rational number and the operations of addition and multiplication on the rationals. The arithmetic of rationals is therefore adequate for the formulation of all problems of elementary number theory.

Since 1948 Julia Robinson has dedicated most of her professional career to the tenth problem on Hilbert's famous list. She worked for short periods as a junior mathematician to RAND (contraction of the term *research and development*) Corporation in Santa Monica (1949-50) and to the Office of Naval Research (1951-52). In the 1950s Julia Robinson continued to undertake research in mathematics, but also became involved with politics. She worked on Adlai Stevenson's presidential campaigns in 1952 and in 1956 (he was defeated each time by Dwight D. Eisenhower), then she worked for the Democratic party for the next six years.

Her early results on the tenth problem took on added importance in 1961 with the publication of a joint paper with Martin Davis and Hilary Putnam [3] in which it was proved that there is no algorithm for deciding whether an exponential Diophantine equation has a solution in natural numbers. The *exponential Diophantine equations* admit, besides addition and multiplication, the operation of exponentiation. The problem for Diophantine equations was solved by Yuri Matijasevich in 1970, and Julia wrote him enthusiastically ([12]): "... now I know it is true, it is beautiful, it is wonderful. ... If you really are 22, I am especially pleased to think that when I first made the conjecture you were a baby and I just had to wait for you to grow up." The work of Matijasevich was strongly based on the ideas of Julia Robinson, who was very close to solve the problem herself.

The professional life of Julia Robinson until 1975 was not the one she deserved. In fact, she had only the low-level position of lecturer at UC Berkeley in Spring 1960, Fall 1962, 1963-1964, Spring and Fall 1966, Fall 1967, 1969-1970, Spring 1975. Lenore Blum, who met Julia at Berkeley, considers in her Review [1] of the book written by Julia's sister [11] that this situation was "totally bizarre". She bitterly says that logic, mathematics and Berkeley together missed out, and perhaps Hilbert's tenth problem would have been solved at Berkeley, if Julia have had a permanent position and her own Ph.D. students.

Julia Robinson suffered health problems in the 1960s having heart surgery, and afterwards her health improved.

In 1976, Saunders Mac Lane proposed her for membership in the National Academy of Sciences (NAS), and Alfred Tarski and Jerzy Neyman, who were old and not well, had both made the trip to Washington, D.C., just so that they could explain the importance of Julia's work. Julia Robinson became the first woman mathematician to be elected to the NAS. Only after she was elected to the NAS, the Berkeley mathematics department offered her a full professorship with the duty of teaching just one-fourth time, so she was professor at UC Berkeley during 1976-1985. In 1983, she became the first woman president of the American Mathematical Society. Her other honors included election to the American Academy of Arts and Sciences, a grant from the MacArthur Foundation, and an honorary degree from Smith College.

Julia developed leukemia in the summer of 1984. She died in 30 of July 1985 at Oakland, California. The paper [12] containing the biography of Julia Robinson was written (in the first person as if by Julia) by her sister, Constance Reid (January 3, 1918 – October 14, 2010), after long discussions during the last month of Julia's life. Later on, it was included, together with three articles previously published by Lisl Gaal, Martin Davis, and Yuri Matijasevich, in [11]. In 1986, Raphael established the *Julia Bowman Robinson Fund* for fellowships for graduate students in mathematics at Berkeley. At San Diego High School, where outstanding alumni are honored and their pictures are put on display in the Media Center (Library) as inspiration for current students, Julia Robinson's picture is exposed on the Wall of Honor since 1995 ([17]). Since

2000, her sister's picture is displayed there too. The discussions with Raphael and Julia allowed Constance to write several biographies of mathematicians, among which that of Hilbert ([10]) and popular books about mathematics, even if she had not studied mathematics. Reid won the Mathematical Association of America's *George Pólya Award* in 1987 for *The Autobiography of Julia Robinson* ([12]). A festival named *Julia Robinson Mathematics Festival* honours every spring from 2007 her legacy and encourage more students to pursue mathematics. It is not a proper contest, offering problems and activities related to one another and getting progressively more difficult, allowing students to work individually or in groups. Julia Robinson Mathematics Festivals have been held across the United States of America including at Stanford, UC Berkeley, and Princeton as well as at Pixar and Google.

After the death of Julia, the Notices of the AMS (November 1985, pp. 739-42) published a number of tributes written by Robinson's colleagues. Elizabeth Scott, who knew Robinson from their days in graduate school together, described her difficulties in securing a position at Berkeley, and her firm but discreet help for women in Mathematics. "Throughout her life Julia stood up for offering opportunities to all students," wrote Scott. "She also encouraged graduate students and young faculty to have more confidence in their real abilities. She felt that women and minority mathematicians especially needed this support, which she provided with spirit yet in a quiet way... She encouraged us to work together so that all women who have the ability and the desire to do mathematical research can have the opportunity to do so."

In 1992, Matijasevich published a personal account of his collaboration with Julia Robinson ([8]), and in 1993 a book in Russian on Hilbert's tenth problem, which was translated in English in the same year with the aid of Martin Davis ([7]). George Csicsery produced and directed a one-hour documentary about Robinson titled *Julia Robinson and Hilbert's Tenth Problem* (2008).

BIBLIOGRAFIE

[1] Blum, L., Review of *Julia, A Life in Mathematics, by Constance Reid*, AMM 105(10)(1998), 964-972

[2] Davis, M., Matijasevich, Y., Robinson, J., Hilbert's tenth problem. Diophantine equations: positive aspects of a negative solution, Proc. Symp. Pure. Math. 28 (1976), 323-378

[3] Davis, M., Putnam, H., Robinson, J., The decision problem for exponential Diophantine equations, Ann. Math. 74 (2) (1961), 425-436

[4] Feferman, S., JULIA BOWMAN ROBINSON 1919–1985, Biographical Memoirs 63 (1994), The National Academy Press, Washington, D.C.

[5] Feferman, S., The Collected Works of Julia Robinson, American Mathematical Society (1996), 27-41

[6] Hilbert, D., Mathematische Probleme. Vortrag, gehalten aus dem internationalen Mathematiker Kongress zu Paris 1900, Nach. K. Ges. Wiss., Göttingen, Math. -Phys. Kl. (1900), 253-297

[7] Matijasevich, Y., Hilbert's tenth problem, The MIT Press, Cambridge, Massachusetts, 1993

[8] Matijasevich, Y., My collaboration with Julia Robinson, Math. Intelligencer 14 (4) (1992), 38-45
[9] Matijasevich, Y., On recursive unsolvability of Hilbert's tenth problem, Proceedings of Fourth International Congress on Logic, Methodology and Philosophy of Science, Bucharest, 1971, Amsterdam: North-Holland (1973), 89-110
[10] Reid, C., Hilbert, Springer-Verlag, New York, 1996
[11] Reid, C., Julia, A Life in Mathematics, Mathematical Association of America, Washington, D.C. (1997)
[12] Reid, C., The autobiography of Julia Robinson, College Math. J. 17 (1986), 3-21
[13] Robinson, J., Definability and decision problems in arithmetic. J. Symb. Logic 14 (1949), 98-114
[14] Robinson, J., Solving Diophantine equations, Proceedings of Fourth International Congress on Logic, Methodology and Philosophy of Science, Bucharest, 1971, Amsterdam: North-Holland (1973), 63-67
[15] Singh, S., Fermat's last theorem: the story of a riddle that confounded the world's greatest minds for 358 years, Fourth Estate, HarperCollins Publishers, EPub Edition 2013
[16] Yandell, B. H., The Honors Class-Hilbert's Problems and Their Solvers, AK Peters, Nattick, Masachusetts, 2002
[17] https://www.sandiegounified.org/schools/san-diego-high-educational-complex/wall-honor

Tiberiu Popoviciu *Institute of Numerical Analysis*
Romanian Academy, Cluj-Napoca
e-mail: `mira@math.ubbcluj.ro`

Primit la redacţie: 30 Noiembrie 2016